ON THE STUDY OF MULTIPLE HYPERGEOMETRIC FUNCTIONS OF HIGHER ORDER

THESIS PRESENTED
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

BY

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Mr. F.U. Khan
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Dear Mr. F.U. Khan,

We are glad to inform you that the paper entitled

"On Appell's polynomial F_2"

written by you, Mr. M.I. Qureshi and Mr. M.A. Pathan has been refereed and accepted for publication.

To the best of our knowledge it will be published in Vol. 15, no.2 of 1984. Thank you for your interest in this journal. We look forward to your submission of your future work.

With best wishes.

Sincerely yours

B. S. Tam

Dr. Bit-Shun Tam
Editorial Committee
Tamkang Journal of Mathematics

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MY FATHER

C E R T I F I C A T E

This is to certify that contents of present thesis entitled " ON THE STUDY OF MULTIPLE HYPERGEOMETRIC FUNCTIONS OF HIGHER ORDER " is an original research work of Mr. Mohammad Idris Qureshi, done under my supervision. A part of this work has already been accepted for publication.

I further certify that the work of this thesis, either partly or fully has not been submitted to any other institution for the award of any other degree.

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Mohd Idris Qureshi

ALIGARH (MOHAMMAD IDRIS QURESHI)

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SUMMARY OF THE THESIS

In a recent book " Multiple hypergeometric functions and applications " of Harold Exton [41], a systematic study of hypergeometric functions of several variables and their applications to Physics, Statistics and variety of other fields were discussed. Since that book was published, a number of advances have been made by various research workers, especially, Buschman and Srivastava [16], Abiodun [1], Chhabra and Rusia [26], Deshpande [28], Khan [54], Khan and Pathan [55], [56], Pathan [79] to [86], Sharma [94], Sharma and Abiodun [95], Srivastava [109], Srivastava and Pathan [113] and Karlsson [53]. This suggests that the theory of multiple hypergeometric functions can be greatly extended and that the resulting theory will have considerable impact on the subjects that originally suggested the extensions.

Multiple hypergeometric functions constitute a natural generalization of the hypergeometric function of one variable. In the present work the author explores the interconnection of double hypergeometric functions of Appell [3], Humbert [49], Horn [48], Kampé de Fériet [4], triple hypergeometric functions of Saran [89], Jain [50], Srivastava [101], [102], Exton [36], Sharma [91], quadruple hypergeometric functions of Exton [37], [39], Pathan [83] and n-ple hypergeometric functions of Carlson [19], Karlsson [51], Pathan [84], Chandel [24], [25], Lauricella [60], Humbert [41, p.42(2.1.1.1)], Erdélyi [41,p.42 (2.1.1.2)], Srivastava and Exton [110], Exton [38], [41, pp.89 (3.4.2), 43 (2.1.1.4,5), 97(3.5.1, 2)]. Their expansions, generating functions, summations, transformations and reductions are studied. We shall obtain few results which may be of some interest and we state the general case in which a pattern is involved. General formulas are developed and their several special cases, supplementary to those in literature are obtained. They do not seem to have appeared in the literature and further, although they are not difficult to obtain, do not seem to be otherwise well known. In some instances, we shall be able to describe a number of well known results as special cases of our findings.

Thus the aim of the present work is two fold. First, we correct some well known results of Manocha and Sharma [70], and Exton [40], [41] which were widely used by many researchers such as Khan [54] in their work. Second, we attempt to give new results which unify and generalize certain results scattered in the literature by a number of workers from time to time. Among them the names of Abiodun [1], Bailey [6], [10], Erdélyi [31], [32], Khan and Pathan [54], [55], Khan [57], [58], Manocha [68], [69], Manocha and Sharma [70], Munot [72], Preece [87], Sharma [94], Srivastava [99], [100], [106], [107], [108], [109], Srivastava and Exton [111], Pathan [81], [86], Appell and Kampé de Fériet [4] and Exton [41] are worth mentioning.

Chapter I contains definitions, notations of various hypergeometric functions with their convergence conditions. This serves two purposes. First, it discusses the basic concepts and the background of the functions. Second, it seeks to place the study of later chapters in such a way that explicit references may be applied and be brought gradually to a level of considerable understanding.

In Chapter II, we obtain a number of linear, bilinear and bilateral generating functions for Jacobi, generalized Laguerre, Ultraspherical, Gegenbauer, Legendre, Tchebicheff, Rainville and Hermite polynomials by making use of series identities, fractional derivatives and Laplace transform technique. Some known generating functions of Manocha [69], Sharma [94] and Khan [57], [58] are deduced as special cases from our main results.

Summation formulas for hypergeometric and product of hypergeometric polynomials were obtained by Manocha and Sharma [70] which are the generalisations of the results of Carlitz [17] and Halim and Salam [47]. In Chapter III, we show that certain summations of Manocha and Sharma [70] are erroneous. These formulas are corrected with the help of fractional derivatives and series manipulations. A wide variety of summations apertaining to hypergeometric functions of Appell, Gauss, Humbert and Kampé de Fériet are obtained by operational methods from our main summations involving Srivastava's triple and Kampé de Fériet's double series. A few known results of Munot [72] involving Jacobi's and Appell's polynomials follow as special cases. Certain double infinite sums involving Appell's F_2 and Jacobi's polynomials are also deduced.

Chapter IV is concerned with the seven erroneous reduction formulae of Exton [40], [41] for Lauricellas's triple hypergeometric functions F_4 and F_{14}. These formulas are corrected. Exton obtained these results by using Laplace integral representation together with some known results. Instead we shall deduce them by series manipulations. Research workers in Special functions may notice that these results of Exton [40], [41] deserve to become more widely used in future and will not be surprised to find a certain emphasis on their corrections and usefulness. Some more linear and quadratic reductions and transformations of F_E into Kampé de Fériet's, Horn's and Srivastava's functions are given.

Chapter V deals with a number of transformation and reduction formulae for Srivastava's triple series $F^{(3)}$. In few cases $F^{(3)}$ is transformed into Lauricella's triple series $F_A^{(3)}$ or their combinations. Our main results are obtained with the help of an integral [81] for a product of three Whittaker's functions and an integral of Olsson [76]. A reduction of Srivastava's $F^{(3)}$ into a combination of four Kampé de Fériet's functions is also given. A number of known results of Bailey [6], [10] and Preece [87] follow as special cases. Results of this chapter are addition to similar known results of Buschman and Srivastava [16] for Kampé de Fériet's function.

Chapter VI is devoted to transformation and reduction formulas of Exton's functions ${}^{(1)}H_4{}^{(3)}$ and ${}^{(1)}H_3{}^{(3)}$ of three variables and K_1, K_2 and K_3 of four variables into functions of Srivastava, Kampé de Fériet, Appell, Horn, Lauricella, Carlson, Pathan and Exton. Some results of Saran's functions F_E, F_F, Appell's F_2, F_3 and Horn's H_3, H_4 are deduced as special cases. The results thus deduced enable us to find the associated variations of earlier known results of Pathan [78], [79], [85], Bhatt [12], Pandey [77], Exton [41], [42], Srivastava [107] and Khan and Pathan [55].

Sections and equations have been numbered chapterwise. A comprehensive bibliography appears at the end with the authors names in alphabetical order. References to the bibliography are numbered in brackets.

This thesis concludes with an appendix which contains reprints of a few published papers and copies of acceptance letters.

The part of our work which has been published or accepted for publication is given below.

1. On some transformation and reduction formulae of generalised Horn function $E_4^{(P)}$ - I, Published in Soochow J. of Mathematical and Natural Sciences, 8 (1982), 165- 170.

2. A note on the reducibility of the triple hypergeometric function F_S, Published in Math. Chronicle, 12 (1983), 129- 133.

3. On Appell's polynomial F_2, Accepted in Tamkang J. Math., 15 (2) (1984).
This paper was also presented at 48th Annual Conference of Indian Mathematical Society (1982), Abstract No.102,p.47.

4. On a reduction formula of Kampé de Fériet's hypergeometric function of higher order, Accepted in Bull. Inst. Math. Acad. Sinica, 12 (2) (1984).

5. On certain combinations of finite sums involving triple hypergeometric series $F^{(3)}$, Accepted in Mathematicae Notae (1984).

6. A note on hypergeometric polynomials, Accepted in J.Austral. Math.Soc. Series B Applied Math. (1983).

7. A note '' on the sum of Appell function F_2'', Accepted for publication in Indian J. Pure and Applied Math.

8. Finite double sums of Kampé de Fériet's hypergeometric function of higher order.
This paper was presented at 49th Annual Conference of Indian Mathematical Society (1983), Abstract No. 138, p 62.

I hope this thesis has presented topics in multiple hypergeometric functions of some interest and has raised some problems worthty of further research. It has been our aim and desire to show that there is a fruitful interaction and connection between the hypergeometric functions of different types and nature, while at the same time demonstrating that multiple hypergeometric series possess a special character and have particular applications which clearly make them not just a subtopic or curious generalisation in the study of hypergeometric functions.

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CHAPTER I

POWER SERIES FORMS OF HYPERGEOMETRIC FUNCTIONS AND THEIR CONVERGENCE CONDITIONS

1.1 INTRODUCTION :

A power series known as Gaussian hypergeometric function, is given by

$$ {}_2F_1\left[\begin{matrix} a,b; & \\ & z \\ c; & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n\,z^n}{(c)_n\;\;n!}, \qquad \ldots(1.1.1)$$

where z is a real or complex variable, a, b and c are parameters which can take arbitrary real or complex values, $c \neq 0, -1, -2, \ldots$ and the Pochhammer symbol $(a)_n$ denotes the quantity in terms of gamma function

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma a} = \begin{cases} 1, & \text{if } n = 0, \\ a\,(a+1)\ldots(a+n-1), & \text{if } n=1,2,3,\ldots . \end{cases}$$

This function is of fundamental importance in the theory of Special functions. The importance lies in the well known fact that almost all of the commonly used functions of applicable Mathematics and Mathematical physics are expressible as its special or confluent cases.

The hypergeometric function (1.1.1) can be generalized by simply increasing the number of numerator, denominator parameters and variables. These generalizationshave been studied by many research workers. This chapter attempts to give a brief account of the basic theory of hypergeometric functions of one and more variables. We have been guided by the goal of a sufficiently detailed exposition of those hypergeometric functions which are of use to our later chapters of the thesis. It has naturally led to a certain curtailment of the purely theoretic part and properties of other types of Special functions. We have always sought the simplest way of defining the hypergeometric functions and deriving their special cases, without concern for historical or other considerations.

1.2 GENERALIZED HYPERGEOMETRIC FUNCTION :

The single hypergeometric function of higher order is defined by

$$
{}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p ; \\ \\ b_1, b_2, \ldots, b_q ; \end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{\left\{\prod_{i=1}^{p} (a_i)_n\right\} z^n}{\left\{\prod_{j=1}^{q} (b_j)_n\right\} n!} , \quad \ldots(1.2.1)
$$

where $(a_i)_n$ is Pochhammer's symbol, given by

$$(a_i)_n = \Gamma(a_i+n) / \Gamma(a_i)$$

and denominator parameters $b_1, b_2, \ldots, b_q$ are neither zero nor negative integers, numerator parameters $a_1, a_2, \ldots, a_p$ may be zero or negative integers.

The convergence conditions of ${}_pF_q$ are given below :

(i) If $p \leq q$, the series converges for all finite z (real or complex) and it is absolutely convergent if $z = 1$.

(ii) If $p = q+1$, the series converges for $|z| < 1$.

(iii) If $p > q+1$, the series only converges when $z = 0$.

(iv) If $p = q+1$, the series is absolutely convergent on the circle $|z| = 1$, that is

$$\text{Re}\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i \right) > 0 \quad \text{for } z = 1,$$

and

$$\text{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_i \right) > -1 \quad \text{for } z = -1.$$

When $p = 2$ and $q = 1$, (1.2.1) reduces to ordinary hypergeometric function (1.1.1) of second order ${}_2F_1$ and was given by

C. F. Gauss in 1812.

When $p = q = 1$, (1.2.1) reduces to confluent hypergeometric function ${}_1F_1$ and was given by E. E. Kummer in 1836.

1.3 APPELL'S FUNCTIONS :

In 1880, Appell [3] has defined the following four double series of second order

$$F_1[a;b,c;d;x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n\, x^m\, y^n}{(d)_{m+n}\quad m!\quad n!}, \quad \ldots(1.3.1)$$

$$F_2[a;b,c;d,e;x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n\, x^m y^n}{(d)_m\,(e)_n\quad m!\, n!}, \quad \ldots(1.3.2)$$

$$F_3[a,b;c,d;e;x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_m(b)_n(c)_m(d)_n x^m y^n}{(e)_{m+n}\quad m!\quad n!}, \ldots(1.3.3)$$

$$F_4[a;b;c,d;x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\,(b)_{m+n}\, x^m y^n}{(c)_m\,(d)_n\, m!\, n!}, \quad \ldots(1.3.4)$$

The regions of convergence of Appell's series are given below :

SERIES	REGION OF CONVERGENCE
F_1	$\|x\| < 1,\ \|y\| < 1$
F_2	$\|x\| + \|y\| < 1$
F_3	$\|x\| < 1,\ \|y\| < 1$
F_4	$\|x\|^{1/2} + \|y\|^{1/2} < 1$

1.4 HUMBERT'S FUNCTIONS :

In 1920, Humbert [49] gave the seven double series. Five of them are given below :

$$\Phi_1\left[a;b;c;x,y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m\,x^m\,y^n}{(c)_{m+n}\,m!\,n!}, \qquad \ldots(1.4.1)$$

$$\Phi_2\left[a,b;c;x,y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_m\,(b)_n\,x^m\,y^n}{(c)_{m+n}\,m!\,n!}, \qquad \ldots(1.4.2)$$

$$\Phi_3\left[a;b;x,y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_m\,x^m\,y^n}{(b)_{m+n}\,m!\,n!}, \qquad \ldots(1.4.3)$$

$$\Psi_1\left[a;b;c,d;x,y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m\,x^m\,y^n}{(c)_m\,(d)_n\,m!\,n!}, \qquad \ldots(1.4.4)$$

$$\Psi_2[a;b,c;x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\, x^m\, y^n}{(b)_m (c)_n\, m!\, n!}, \qquad \ldots(1.4.5)$$

For Φ_1 and Ψ_1, we need $|x| < 1$ for convergence, the convergence is otherwise unrestricted. (see [15], p.118 (first line)).

1.5 HORN'S FUNCTIONS :

During 1889 to 1939, Horn [48] has been defined many double hypergeometric series of second order. Four of them are given below :

$$G_1[a,b,c;x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{n-m}(c)_{m-n}\, x^m y^n}{m!\, n!}, \qquad \ldots(1.5.1)$$

$$H_1[a,b,c,d;x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_{m+n}(c)_n\, x^m y^n}{(d)_m\, m!\, n!}, \qquad \ldots(1.5.2)$$

$$H_3[a,b;c;x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}\,(b)_n\, x^m\, y^n}{(c)_{m+n}\, m!\, n!}, \qquad \ldots(1.5.3)$$

$$H_4[a,b;c,d;x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(b)_n\, x^m\, y^n}{(c)_m\, (d)_n\, m!\, n!}. \qquad \ldots(1.5.4)$$

The regions of convergence of these functions are given below :

SERIES	CARTESIAN EQUATION
G_1	$r + s = 1$
H_2	$4rs = (s-1)^2$
H_3	$r + (s-\frac{1}{2})^2 = \frac{1}{4}$
H_4	$4r = (s-1)^2$

where the positive quantities r, s are the associated radii of convergence of the concerned double power series such that $|x| < r$ and $|y| < s$.

1.6 KAMPÉ DE FÉRIET'S FUNCTION :

Let $F^{A:B;D}_{E:G;H}$ denotes the Kampé de Fériet's double hypergeometric function [4, p.150] in the modified notation of Srivastava and Panda [112, pp.423(26), 424(27); see also 113, p.23(1.2,1.3)], defined by

$$F^{A:B;D}_{E:G;H}\left[\begin{matrix}(a_A) : (b_B); (d_D) ; \\ (e_E) : (g_G); (h_H) ;\end{matrix} x, y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{[(a_A)]_{m+n} \, [(b_B)]_m \, [(d_D)]_n \, x^m \, y^n}{[(e_E)]_{m+n} \, [(g_G)]_m \, [(h_H)]_n \, m! \, n!}, \qquad \ldots(1.6.1)$$

where for the sake of convenience (a_A) abbreviates the array of A parameters $a_1, a_2, \ldots, a_A$ and $[(b_B)]_m = \prod_{i=1}^{B} (b_i)_m$, with similar interpretations for (e_E), (g_G), et cetera and for convergence of the double hypergeometric series,

(i) $A+B \leq E+G$, $A+D \leq E+H$ and $\max\{|x|, |y|\} < \infty$,

or

(ii) $A+B = E+G+1$, $A+D = E+H+1$ and

$$\begin{cases} |x|^{1/(A-E)} + |y|^{1/(A-E)} < 1, \text{ if } A > E \\ \max\{|x|, |y|\} < 1, \text{ if } A \leq E . \end{cases}$$

This function was introduced in 1921.

1.7 LAURICELLA'S FUNCTIONS :

In 1893, Lauricella [60, pp.113-114] gave the fourteen hypergeometric functions, out of which three n-ple series are given below in his notations

$$F_A^{(n)} [a; b_1, b_2, \ldots, b_n ; c_1, c_2, \ldots, c_n; x_1, x_2, \ldots, x_n]$$
$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1+m_2+\ldots+m_n} (b_1)_{m_1} (b_2)_{m_2} \ldots (b_n)_{m_n} x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}}{(c_1)_{m_1} (c_2)_{m_2} \ldots (c_n)_{m_n} \, m_1! \ldots m_n!} ,$$
...(1.7.1)

$$F_C^{(n)}\left[a;\ b;\ c_1,\dots,\ c_n;\ x_1,\dots,\ x_n\right]$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n}(b)_{m_1+\dots+m_n}\, x_1^{m_1}\dots x_n^{m_n}}{(c_1)_{m_1}\dots(c_n)_{m_n}\ m_1!\dots\ m_n!}, \quad \dots(1.7.2)$$

$$F_D^{(n)}\left[a;\ b_1,\dots,\ b_n;\ c;\ x_1,\dots,\ x_n\right]$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n}(b_1)_{m_1}\dots(b_n)_{m_n}\ x_1^{m_1}\dots x_n^{m_n}}{(c)_{m_1+\dots+m_n}\ m_1!\dots m_n!}, \quad \dots(1.7.3)$$

The regions of convergence of above series are given by means of the following table :

SERIES	REGION OF CONVERGENCE
$F_A^{(n)}$	$\lvert x_1\rvert + \dots + \lvert x_n\rvert < 1$
$F_C^{(n)}$	$\lvert x_1\rvert^{1/2} + \dots + \lvert x_n\rvert^{1/2} < 1$
$F_D^{(n)}$	$\lvert x_1\rvert < 1, \dots, \lvert x_n\rvert < 1$

The remaining Lauricella's triple hypergeometric functions of second order were given in Saran's revised notations [89]. Four of them are given below :

$$F_E \left[a, a, a; b, c, c; d, e, f; x, y, z \right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p} (b)_m (c)_{n+p} x^m y^n z^p}{(d)_m (e)_n (f)_p m! n! p!}, \qquad \ldots(1.7.4)$$

$$F_F \left[a, a, a; b, c, b; d, e, e; x, y, z \right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p} (b)_{m+p} (c)_n x^m y^n z^p}{(d)_m (e)_{n+p} m! n! p!}, \qquad \ldots(1.7.5)$$

$$F_G \left[a, a, a; b, c, d; e, f, f; x, y, z \right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p} (b)_m (c)_n (d)_p x^m y^n z^p}{(e)_m (f)_{n+p} m! n! p!}, \qquad \ldots(1.7.6)$$

$$F_S \left[a, b, b; c, d, e; f, f, f; x, y, z \right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_m (b)_{n+p} (c)_m (d)_n (e)_p x^m y^n z^p}{(f)_{m+n+p} m! n! p!}, \qquad \ldots(1.7.7)$$

In 1974, Karlsson [52,p.243] gave the following regions of convergence for the above mentioned series. Although the regions of convergence of different triple series have been studied by many Mathematicians like Saran [90], Srivastava [98], Dhawan [30] and Pandey [77], which have been given either incorrectly or incompletely.

NUMBERING IN LAURICELLA'S CONJECTURE	SARAN'S REVISED NOTATION	CARTESIAN EQUATION (S)
F_4	F_E	$r+(s^{1/2}+t^{1/2})^2 = 1$
F_{14}	F_F	$o < r < (1-s)^2 : t^{1/2}+r^{1/2} = 1$ $(1-s)^2 \leq r < 1-s : \dfrac{r}{1-s}+\dfrac{t}{s} = 1$
F_8	F_G	$r+t = 1,\ r+s = 1$
F_7	F_S	$r=1,\ s = 1,\ t = 1$

where r, s and t are associated radii of convergence such that $|x| < r$, $|y| < s$ and $|z| < t$.

1.8 CARLSON'S FUNCTION :

In 1963, Carlson [19, p.453(2.1)] gave the following n-ple hypergeometric function in the form

$$R\left[a; b_1,\ldots, b_n; x_1,\ldots, x_n\right]$$
$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}(1-x_1)^{m_1}\ldots(1-x_n)^{m_n}}{(b_1+b_2+\ldots+b_n)_{m_1+\ldots+m_n}\; m_1!\, m_2!\ldots m_n!} \, . \qquad \ldots(1.8.1)$$

if $|(1-x_i)| < 1$, $(1 \leq i \leq n)$, and by its analytical continuation if $|\arg x_i| < \pi$.

1.9 JAIN'S FUNCTION :

In 1966, Jain [50] gave many confluent forms of Lauricella's triple hypergeometric functions. One of them is given in the form [50, p. 396 (2.9)]

$$S = {}_3\Phi_D^{(1)} \left[a, b, c; d; x, y, z \right]$$
$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p} (b)_m (c)_n x^m y^n z^p}{(d)_{m+n+p} \; m! \, n! \, p!} . \qquad \ldots(1.9.1)$$

The above series (1.9.1) in Exton's notation [36,p.81], is given by

$$S = F_{D_1} \left[a,a,a; b,c,-; d,d,d; x, y, z \right]. \qquad \ldots(1.9.2)$$

1.10 SRIVASTAVA'S FUNCTIONS :

In 1964, Srivastava [98] defined two triple series. One of them is given in the form

$$H_B \left[a, b, c; d, e, f; x, y, z \right]$$
$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+p} (b)_{m+n} (c)_{n+p} x^m y^n z^p}{(d)_m (e)_n (f)_p \; m! \, n! \, p!} . \qquad \ldots(1.10.1)$$

The region of convergence of above series is given in the form:

$r+s+t+2\,(rst)^{1/2} = 1$ such that $|x| < r$, $|y| < s$ and $|z| < t$.

In 1967, a unification of Lauricella's fourteen triple hypergeometric functions $F_1, \ldots, F_{14}$ [60, pp.113-114], extended F_R function of Sharma [91, p.613(2)] and three additional functions H_A, H_B and H_C of Srivastava [101, pp.99-100; see also 98 and 105] was given by Srivastava [102, p.428] in the form

$$F^{(3)}\left[\begin{matrix}(a_A)::\ (b_B);(b'_{B'});(b''_{B''}):(c_C);(c'_{C'});(c''_{C''}); \\ (d_D)::\ (e_E);(e'_{E'});(e''_{E''}):(g_G);(g'_{G'});(g''_{G''});\end{matrix}\ x,y,z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{[(a_A)]_{m+n+p}\ [(b_B)]_{m+n}\ [(b'_{B'})]_{n+p}\ [(b''_{B''})]_{p+m}\ [(c_C)]_m}{[(d_D)]_{m+n+p}\ [(e_E)]_{m+n}\ [(e'_{E'})]_{n+p}\ [(e''_{E''})]_{p+m}\ [(g_G)]_m}$$

$$\frac{[(c'_{C'})]_n\ [(c''_{C''})]_p\ x^m\ y^n\ z^p}{[(g'_{G'})]_n\ [(g''_{G''})]_p\ m!\,n!\,p!}\ . \qquad \ldots(1.10.2)$$

The region of convergence of above triple power series is given in the literature [26, p.156; see also 28, p. 40].

$A+B+B''+C \leq D+E+E''+G$, $A+B+B'+C' \leq D+E+E'+G'$, $A+B'+B''+C'' \leq D+E'+E''+G''$, and $|x|<1$, $|y|<1$, $|z|<1$; but if $A+B+B''+C = D+E+E''+G+1$, $A+B+B'+C' = D+E+E'+G'+1$, $A+B'+B''+C'' = D+E'+E''+G''+1$ then $|x|$, $|y|$ and $|z|$ are to be restricted appropriately, so that the series involved are either terminating or convergent.

1.11 SRIVASTAVA AND EXTON FUNCTION :

In 1973, Srivastava and Exton [110, p.373(12,13)] gave the following n-ple series

$$\Phi_D^{(n)}\left[a, b_1, \ldots, b_{n-1}, -; c; x_1, \ldots, x_n\right]$$
$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_{n-1})_{m_{n-1}}}{(c)_{m_1+\ldots+m_n}} \frac{x_1^{m_1}\ldots x_n^{m_n}}{m_1!\ldots m_n!}. \quad \ldots(1.11.1)$$

For region of convergence of above series, we have

$|x_1|<1,\ldots,|x_{n-1}|<1$; x_n may take any finite value.

1.12 EXTON'S FUNCTIONS :

In 1972, Exton [37; see also 39] gave 21 quadruple hypergeometric functions. Four of them in the form of explicit series expansions are given below :

$K_1 [a,a,a,a;b,b,b,c;d,e,f,d;x,y,z,t]$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}\,(b)_{m+n+p}\,(c)_q\, x^m\, y^n\, z^p\, t^q}{(d)_{m+q}\,(e)_n\,(f)_p\; m!\, n!\, p!\, q!}, \qquad \ldots(1.12.1)$$

$K_2 [a,a,a,a; b,b,b,c;d,e,f,g; x, y, z, t]$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}\,(b)_{m+n+p}\,(c)_q\; x^m\, y^n\, z^p\, t^q}{(d)_m\,(e)_n\,(f)_p\,(g)_q\; m!\, n!\, p!\, q!}, \qquad \ldots(1.12.2)$$

$K_3 [a,a,a,a; b,b,c,c; d, e, e, d; x, y, z, t]$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}\,(b)_{m+n}\,(c)_{p+q}\, x^m\, y^n\, z^p\, t^q}{(d)_{m+q}\,(e)_{n+p}\; m!\, n!\, p!\, q!}, \qquad \ldots(1.12.3)$$

$K_{10} [a,a,a,a; b,b, c,d; e, f, g, h; x, y, z, t]$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}\,(b)_{m+n}\,(c)_p\,(d)_q\, x^m y^n\, z^p\, t^q}{(e)_m\,(f)_n\,(g)_p\,(h)_q\; m!\, n!\, p!\, q!}, \qquad \ldots(1.12.4)$$

The regions of convergence of above quadruple series, may be investigated by means of the general theory of convergence given in the book of Exton [41, p.65].

Exton included the following three more n-ple series in his recent book [41, pp. 43(2.1.1.5), 97(3.5.1,2)]

$$\Phi_3^{(n)}\left[b_1,\ldots,b_{n-1};\ c;\ x_1,\ldots,x_n\right]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}\ x_1^{m_1}\cdots x_n^{m_n}}{(c)_{m_1+\cdots+m_n}\ m_1!\cdots m_n!} \qquad \ldots(1.12.5)$$

and converges for all finite values of $x_1,\ldots,x_n$;

$${}^{(k)}H_3^{(n)}\left[a,\ b_{k+1},\ldots,b_n;\ c;\ x_1,\ldots,x_n\right]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{2m_1+2m_2+\cdots+2m_k+m_{k+1}+\cdots+m_n}(b_{k+1})_{m_{k+1}}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \cdot \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}, \qquad \ldots(1.12.6)$$

$${}^{(k)}H_4^{(n)}\left[a,\ b_{k+1},\ldots,b_n;\ c_1,\ldots,c_n;\ x_1,\ldots,x_n\right]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{2m_1+2m_2+\cdots+2m_k+m_{k+1}+\cdots+m_n}(b_{k+1})_{m_{k+1}}\cdots(b_n)_{m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}} \cdot \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}. \qquad \ldots(1.12.7)$$

The regions of convergence of above two power series (1.12.6), (1.12.7) are given by Karlsson [53, p.73] in 1978, in the following form

SERIES	REGION OF CONVERGENCE
$(k)_{E_3}(n)$	$\max\left\{\lvert x_{k+1}\rvert, \lvert x_{k+2}\rvert, \ldots, \lvert x_n\rvert\right\} < \frac{1}{2} + \left\{\frac{1}{4} - (\lvert x_1\rvert + \ldots + \lvert x_k\rvert)\right\}^{1/2}$
$(h)_{E_4}(n)$	$2\left\{\lvert x_1\rvert^{1/2} + \ldots + \lvert x_k\rvert^{1/2}\right\} + \lvert x_{k+1}\rvert + \ldots + \lvert x_n\rvert < 1$

1.13 PATHAN'S FUNCTION :

In 1979, a general quadruple hypergeometric series $F_P^{(4)}$ was considered by Pathan [83, p.172(1.2); see also 84, p.51(1)] in the form

$$F_P^{(4)}\left[\begin{matrix}(a_A)::(b_B);(d_D);(e_E);(g_G):(h_H);(k_K);(m_M);(n_N);\\ \\ (a'_{A'})::(b'_{B'});(d'_{D'});(e'_{E'});(g'_{G'}):(h'_{H'});(k'_{K'});(m'_{M'});(n'_{N'});\end{matrix}\; x, y, z, u\right]$$

$$= \sum_{q,r,s,j=0}^{\infty} \frac{[(a_A)]_{q+r+s+j}\,[(b_B)]_{q+r+s}\,[(d_D)]_{r+s+j}}{[(a'_{A'})]_{q+r+s+j}\,[(b'_{B'})]_{q+r+s}\,[(d'_{D'})]_{r+s+j}}\,.$$

$$\cdot \frac{[(e_E)]_{s+j+q}\,[(g_G)]_{j+q+r}\,[(h_H)]_q\,[(k_K)]_r\,[(m_M)]_s\,[(n_N)]_j}{[(e'_{E'})]_{s+j+q}\,[(g'_{G'})]_{j+q+n}\,[(h'_{H'})]_q\,[(k'_{K'})]_r\,[(m'_{M'})]_s\,[(n'_{N'})]_j}$$

$$\frac{x^q\, y^r\, z^s\, u^j}{q!\, r!\, s!\, j!} \cdot \qquad \ldots(1.13.1)$$

It being understood that $|x|$, $|y|$, $|z|$ and $|u|$ are sufficiently small to ensure the convergence of the concerned quadruple series.

The following reducible cases of $F_P^{(4)}$ are obvious :

(i) For x or y or z or u = o, or any one of the numerator parameters (h_H), (k_K), (m_M), (n_N) is zero, $F_P^{(4)}$ reduces to a general class of the triple hypergeometric series $F^{(3)}$ of Srivastava, given by (1.10.2).

(ii) For D = E = G = H = K = M = A' = B' = D' = E' = G' = o and A = B = N = H' = K' = M' = N' = 1 , $F_P^{(4)}$ reduces to Exton's function $K_2[a_1, a_1, a_1, a_1; b_1, b_1, b_1, n_1; h'_1, k'_1, m'_1, n'_1; x, y, z, u]$ given by the equation (1.12.2) and is related to Chandel's function [41, p.91]

$${}^{(3)}_{(1)}E_C^{(4)}\left[b_1, n_1, a_1; h'_1, k'_1, m'_1, n'_1;\ x,\ y,\ z,\ u\right].$$

(iii) Similarly by suitable adjustment of parameters and variables in $F_P^{(4)}$, we can easily observe that $F_P^{(4)}$ is a unification of all quadruple hypergeometric functions of Chandel's ${}^{(0)}_{(1)}E_C^{(4)}$, ${}^{(1)}_{(1)}E_C^{(4)}$, ${}^{(3)}_{(1)}E_C^{(4)}$ [24, p.120(2.3); see also 25, p.177], Exton's ${}^{(0)}_{(1)}E_D^{(4)}$, ${}^{(1)}_{(1)}E_D^{(4)}$, ${}^{(3)}_{(1)}E_D^{(4)}$ [38] ${}^{(0)}_{(2)}E_D^{(4)}$, ${}^{(1)}_{(2)}E_D^{(4)}$, ${}^{(3)}_{(2)}E_D^{(4)}$ [41, p.89(3.4.2)], K_{11}, K_{15} [41, p.78(3.3.11,15)], $\Xi_1^{(4)}$ [41, p.43(2.1.1.4)], $\Phi_3^{(4)}$ given by the equation (1.12.5), Karlsson's $F^{1:2}_{1:1}$ [51, p.265(1)], Lauricella's $F_A^{(4)}$, $F_B^{(4)}$, $F_C^{(4)}$, $F_D^{(4)}$ [60, pp113-114], Carlson's R given by the equation (1.8.1), Humbert's $\Psi_2^{(4)}$ [41, p.42(2.1.1.1)], Erdélyi's $\Phi_2^{(4)}$ [41, p. 42 (2.1.1.2)] and Srivastava-Exton's $\phi_D^{(4)}$ given by the equation (1.11.1).

CHAPTER II

GENERATING FUNCTIONS

2.1 INTRODUCTION :

Let c_n; $n = o,1,2,\ldots$, be a specified sequence independent of x and t. We say that $G(x,t)$ is a generating function of the set $g_n(x)$ if

$$G(x,t) = \sum_{n=o}^{\infty} c_n \; g_n(x) \; t^n . \qquad \ldots(2.1.1)$$

The purpose of this chapter is to obtain some linear, bilinear and bilateral generating functions of different kinds of hypergeometric polynomials.

In section 2.2, linear generating functions for even and odd hypergeometric polynomials are obtained by making use of series identities. By the process of confluence and Laplace transform technique, we derive some more linear generating functions for Jacobi, generalized Laguerre, Ultraspherical, Gegenbauer, Legendre, Tchebicheff, Rainville and Hermite polynomials. Some known generating functions of Manocha [69] and Sharma [94] are also obtained as special cases.

In section 2.3, a bilinear generating function for Jacobi polynomials is obtained by means of the fractional derivatives. A number of interesting linear, bilinear and bilateral generating functions for Jacobi and Laguerre polynomials are derived as special cases from our main generating function. Known generating functions of Khan [57], [58] are also deduced as special cases.

2.2 USE OF SERIES IDENTITIES :

The idea of separation of a power series into its even and odd terms, exhibited by the elementary identity ,

$$\sum_{n=0}^{\infty} A(n) = \sum_{n=0}^{\infty} A(2n) + \sum_{n=0}^{\infty} A(2n+1), \qquad \ldots(2.2.1)$$

is atleast as old, as the series themselves.

By using (2.2.1) in the generalised hypergeometric series (1.2.1), Barr [11, p.591(1)] obtained the following hypergeometric series identity

$${}_pF_q\left[\begin{matrix}(a_p); \\ (b_q);\end{matrix}\; z\right] = {}_{2p}F_{2q+1}\left[\begin{matrix}\frac{(a_p)}{2}, \frac{(a_p)+1}{2}; \\ \frac{1}{2}, \frac{(b_q)}{2}, \frac{(b_q)+1}{2};\end{matrix}\; 4^{(p-q-1)}\, z^2\right]$$

$$+\frac{z\prod_{i=1}^{p}a_i}{\prod_{j=1}^{q}b_j}\;{}_{2p}F_{2q+1}\left[\begin{matrix}\frac{(a_p)+1}{2},\ \frac{(a_p)+2}{2}\ ; \\ \frac{3}{2},\frac{(b_q)+1}{2},\ \frac{(b_q)+2}{2}\ ;\end{matrix}\ 4^{(p-q-1)}\,z^2\right]. \quad \ldots(2.2.2)$$

It is to be noted that the identity (2.2.2) was also obtained by Carlson [20, p.254(10)], Manocha [69,p.43(3)], Sharma [93, p. 95(1); see also 92] and Srivastava [109,p.191(3)].

The main object of the present section is to show how these extra ordinary identities (2.2.1) and (2.2.2) can be used to obtain the linear generating functions for different kinds of hypergeometric polynomials.

Feldheim [44, p.120(12)] obtained a generating function for Jacobi's polynomial [88, p.254(1)] of order a, b and degree n in x,

$$P_n^{(a,b)}(x) = \frac{(1+a)_n}{n!}\;{}_2F_1\left[\begin{matrix}-n,\ n+1+a+b\ ; \\ 1+a\ ;\end{matrix}\ \frac{1-x}{2}\right] \quad \ldots(2.2.3)$$

in the form

$$\sum_{n=0}^{\infty}\frac{t^n}{(1+a)_n}\,P_n^{(a,b-n)}(x) = \exp\left\{\frac{t(1+x)}{2}\right\}\,{}_1F_1\left[\begin{matrix}-b; \\ 1+a;\end{matrix}\ \frac{t(1-x)}{2}\right]. \quad \ldots(2.2.4)$$

where $Re(a) > -1$ and $|t| < 1$.

By (2.2.1) and (2.2.3), we rewrite the above equation in the form

$$\sum_{n=0}^{\infty} \frac{t^{2n}}{(1+a)_{2n}} P_{2n}^{(a,b-2n)}(x) + \frac{t}{(1+a)} \sum_{n=0}^{\infty} \frac{t^{2n}}{(2+a)_{2n}} P_{2n+1}^{(a,b-2n-1)}(x)$$

$$= \exp\left\{\frac{t(1+x)}{2}\right\} \left\{ {}_2F_3\left[\begin{matrix}\frac{-b}{2}, \frac{-b+1}{2}; \\ \frac{1}{2}, \frac{1+a}{2}, \frac{2+a}{2};\end{matrix} \frac{t^2(1-x)^2}{16}\right] - \frac{bt(1-x)}{2(1+a)} \right.$$

$$\left. {}_2F_3\left[\begin{matrix}\frac{-b+1}{2}, \frac{-b+2}{2}; \\ \frac{3}{2}, \frac{2+a}{2}, \frac{3+a}{2};\end{matrix} \frac{t^2(1-x)^2}{16}\right]\right\}. \qquad \ldots(2.2.5)$$

Changing t into $i\, t^{1/2}$ where $i^2 = -1$, in (2.2.5) and using the fact that

$$e^{ix} = \cos x + i \sin x, \qquad \ldots(2.2.6)$$

we get

$$\sum_{n=0}^{\infty} \frac{(-t)^n}{(1+a)_{2n}} P_{2n}^{(a,b-2n)}(x) + \frac{i\, t^{1/2}}{(1+a)},$$

$$\sum_{n=0}^{\infty} \frac{(-t)^n}{(2+a)_{2n}} P_{2n+1}^{(a,b-2n-1)}(x)$$

$$= \operatorname{Cos}\left\{t^{1/2}\left(\frac{1+x}{2}\right)\right\} \; {}_2F_3\left[\begin{matrix}\frac{-b}{2}, \frac{-b+1}{2} \; ; \\ \frac{1}{2}, \frac{1+a}{2}, \frac{2+a}{2} \; ;\end{matrix} \; \frac{-t(1-x)^2}{16}\right]$$

$$+ \frac{b t^{1/2}(1-x)}{2(1+a)} \operatorname{Sin}\left\{t^{1/2}\left(\frac{1+x}{2}\right)\right\} \; {}_2F_3\left[\begin{matrix}\frac{-b+1}{2}, \frac{-b+2}{2} \; ; \\ \frac{3}{2}, \frac{2+a}{2}, \frac{3+a}{2} \; ;\end{matrix} \; \frac{-t(1-x)^2}{16}\right]$$

$$+i \operatorname{Sin}\left\{t^{1/2}\left(\frac{1+x}{2}\right)\right\} \; {}_2F_3\left[\begin{matrix}\frac{-b}{2}, \frac{-b+1}{2} \; ; \\ \frac{1}{2}, \frac{1+a}{2}, \frac{2+a}{2} \; ;\end{matrix} \; \frac{-t(1-x)^2}{16}\right]$$

$$\frac{i\, t^{1/2}\, b(1-x)}{2(1+a)} \operatorname{Cos}\left\{t^{1/2}\left(\frac{1+x}{2}\right)\right\} \; {}_2F_3\left[\begin{matrix}\frac{-b+1}{2}, \frac{-b+2}{2} \; ; \\ \frac{3}{2}, \frac{2+a}{2}, \frac{3+a}{2} \; ;\end{matrix} \; \frac{-t(1-x)^2}{16}\right].$$

$$\ldots(2.2.7)$$

Now equating the real and imaginary parts in (2.2.7) and again replacing t by $-t$, we get

$$\sum_{n=0}^{\infty} \frac{t^n}{(1+a)_{2n}} P_{2n}^{(a,b-2n)}(x) = \cosh\left\{t^{1/2}\left(\frac{1+x}{2}\right)\right\}$$

$${}_2F_3\left[\begin{matrix}\frac{-b}{2}, \frac{-b+1}{2} \; ; \\ \frac{1}{2}, \frac{1+a}{2}, \frac{2+a}{2} \; ;\end{matrix} \; \frac{t(1-x)^2}{16}\right] - \frac{b\, t^{1/2}(1-x)}{2(1+a)} \sinh\left\{t^{1/2}\left(\frac{1+x}{2}\right)\right\}$$

$${}_2F_3\left[\begin{matrix}\frac{-b+1}{2}, \frac{-b+2}{2} \; ; \\ \frac{3}{2}, \frac{2+a}{2}, \frac{3+a}{2} \; ;\end{matrix} \; \frac{t(1-x)^2}{16}\right]. \qquad \ldots(2.2.8)$$

$$\sum_{n=0}^{\infty} \frac{t^n}{(2+a)_{2n}} P_{2n+1}^{(a,b-2n-1)}(x) = (1+a) \quad t^{-\frac{1}{2}}$$

$$\sin h \left\{ t^{1/2}\left(\frac{1+x}{2}\right) \right\} {}_2F_3 \left[\begin{matrix} \frac{-b}{2}, \frac{-b+1}{2} ; \\ \frac{1}{2}, \frac{1+a}{2}, \frac{2+a}{2} ; \end{matrix} \; \frac{t(1-x)^2}{16} \right] - \frac{b(1-x)}{2}$$

$$\cos h \left\{ t^{1/2}\left(\frac{1+x}{2}\right) \right\} {}_2F_3 \left[\begin{matrix} \frac{-b+1}{2}, \frac{-b+2}{2} ; \\ \frac{3}{2}, \frac{2+a}{2}, \frac{3+a}{2} ; \end{matrix} \; \frac{t(1-x)^2}{16} \right] . \quad \ldots(2.2.9)$$

Replacing x by $(1-\frac{2x}{b})$ in (2.2.8) and (2.2.9), taking $b \to \infty$, using the confluence principle [63, p.48 (§3.5)] in conjugation with

$$\lim_{b \to \infty} \frac{(b)_n}{b^n} = 1 \qquad \ldots(2.2.10)$$

and a known result [5, pp.333-334 (5.6); see also 114, p.105 (5.3.4)]

$$\lim_{b \to \infty} P_n^{(a,b)}\left(1 - \frac{2x}{b}\right)$$

$$= \lim_{b \to \infty} (-1)^n P_n^{(b,a)}\left(\frac{2x}{b} - 1\right)$$

$$= L_n^{(a)}(x) , \qquad \ldots(2.2.11)$$

we get the following two known generating functions of Manocha [69 , p.47(10,11)], respectively.

$$\sum_{n=0}^{\infty} \frac{t^n}{(1+a)_{2n}} L_{2n}^{(a)}(x) = \cosh (t^{1/2})\, {}_0F_3\left[\begin{matrix} \text{———} ; \\ \frac{1}{2}, \frac{1+a}{2}, \frac{2+a}{2} ; \end{matrix} \frac{tx^2}{16}\right]$$

$$- \frac{x\, t^{1/2}}{(1+a)} \sinh (t^{1/2})\, {}_0F_3\left[\begin{matrix} \text{———} ; \\ \frac{3}{2}, \frac{2+a}{2}, \frac{3+a}{2} ; \end{matrix} \frac{tx^2}{16}\right] \qquad \ldots(2.2.12)$$

and

$$\sum_{n=0}^{\infty} \frac{t^n}{(2+a)_{2n}} L_{2n+1}^{(a)}(x) = (1+a)\, t^{-1/2} \sinh (t^{1/2})$$

$${}_0F_3\left[\begin{matrix} \text{———} ; \\ \frac{1}{2}, \frac{1+a}{2}, \frac{2+a}{2} ; \end{matrix} \frac{tx^2}{16}\right] - x \cosh (t^{1/2})\, {}_0F_3\left[\begin{matrix} \text{———} ; \\ \frac{3}{2}, \frac{2+a}{2}, \frac{3+a}{2} ; \end{matrix} \frac{tx^2}{16}\right].$$

$$\ldots(2.2.13)$$

where $L_n^{(a)}(x)$ is generalised Laguerre's polynomial [88, p.200(1)] of order a and degree n in x, defined as

$$L_n^{(a)}(x) = \frac{(1+a)_n}{n!}\, {}_1F_1\left[\begin{matrix} -n ; \\ 1+a ; \end{matrix} x\right]. \qquad \ldots(2.2.14)$$

Similarly by the process of confluence in (2.2.8) and (2.2.9), we get the two more generating functions, respectively.

$$\sum_{n=0}^{\infty} t^n L_{2n}^{(b-2n)}(x) = \operatorname{Cosh}(x\, t^{1/2})\ {}_2F_1\left[\begin{matrix} \frac{-b}{2}, \frac{-b+1}{2} ; \\ \frac{1}{2} ; \end{matrix}\ t\right]$$

$$- b\, t^{1/2} \operatorname{Sinh}(x t^{1/2})\ {}_2F_1\left[\begin{matrix} \frac{-b+1}{2}, \frac{-b+2}{2} ; \\ \frac{3}{2} ; \end{matrix}\ t\right] \qquad \ldots(2.2.15)$$

and

$$\sum_{n=0}^{\infty} t^n L_{2n+1}^{(b-2n-1)}(x) = b \operatorname{Cosh}(x t^{1/2})$$

$${}_2F_1\left[\begin{matrix} \frac{-b+1}{2}, \frac{-b+2}{2} ; \\ \frac{3}{2} ; \end{matrix}\ t\right] - t^{-\frac{1}{2}} \operatorname{Sinh}(x t^{1/2})\ {}_2F_1\left[\begin{matrix} \frac{-b}{2}, \frac{-b+1}{2}, \\ \frac{1}{2} ; \end{matrix}\ t\right].$$

$$\ldots(2.2.16)$$

Again using the hypergeometric series identity (2.2.2) in the exponential term of (2.2.4), we get

$$\sum_{n=0}^{\infty} \frac{t^n}{(1+a)_n} P_n^{(a,b-n)}(x) = {}_0F_1\left[\begin{matrix} \text{———} ; \\ \frac{1}{2} ; \end{matrix}\ \frac{t^2(1+x)^2}{16}\right]$$

$${}_1F_1\left[\begin{matrix} -b ; \\ 1+a ; \end{matrix}\ \frac{t(1-x)}{2}\right] + \frac{t(1+x)}{2}\ {}_0F_1\left[\begin{matrix} \text{———} ; \\ \frac{3}{2} ; \end{matrix}\ \frac{t^2(1+x)^2}{16}\right]$$

$${}_1F_1\left[\begin{matrix} -b ; \\ 1+a ; \end{matrix}\ \frac{t(1-x)}{2}\right]. \qquad \ldots(2.2.17)$$

Now replacing t by ut in (2.2.17), multiplying both the sides by $e^{-u}\, u^{g-1}$, expressing each hypergeometric functions ${}_0F_1$ and ${}_1F_1$ in power series form (1.2.1), integrating with respect to u between the limits o and ∞ and using the integral [88, p.9] for Gamma function

$$I_1 = \int_0^\infty e^{-su}\, u^{G-1}\, du = \frac{\Gamma G}{s^G}\,;\ Re(G) > o,\ Re(s) > o \qquad \ldots(2.2.18)$$

and interpreting the result by using the definition (1.5.4) of Horn's function H_4, we find the following generating function as a combination of two Horn's functions

$$\sum_{n=0}^{\infty} \frac{(g)_n t^n}{(1+a)_n} P_n^{(a,b-n)}(x) = H_4\left[g,\ -b;\ \frac{1}{2},\ 1+a;\ \frac{t^2(1+x)^2}{16},\ \frac{t(1-x)}{2}\right] + \frac{gt(1+x)}{2}\ H_4\left[g+1,\ -b;\ \frac{3}{2},\ 1+a;\ \frac{t^2(1+x)^2}{16},\ \frac{t(1-x)}{2}\right].$$
$$\ldots(2.2.19)$$

Similarly, using a generating function of Manocha [67, p.687(1.4); see also 66, p.457(1.5)]

$$\sum_{n=0}^{\infty} \frac{t^n}{(-a-b)_n} P_n^{(a-n,b-n)}(x) = \exp\left\{\frac{t(1-x)}{2}\right\} {}_1F_1\left[\begin{matrix}-a; \\ -a-b;\end{matrix}\ -t\right]$$
$$\ldots(2.2.20)$$

and applying the same process as in (2.2.19),(2.2.20) gives

$$\sum_{n=0}^{\infty} \frac{t^n (g)_n}{(-a-b)_n} P_n^{(a-n,\, b-n)}(x)$$

$$= H_4 \left[g,\ -a\ ;\ \tfrac{1}{2}\ ,\ -a-b\ ;\ \frac{(1-x)^2 t^2}{16}\ ,\ -t \right] + \frac{(1-x)t\ g}{2}$$

$$H_4 \left[g+1,\ -a;\ \tfrac{3}{2}\ ,\ -a-b;\ \frac{(1-x)^2 t^2}{16}\ ,\ -t \right] . \qquad \ldots(2.2.21)$$

The generating functions (2.2.20) and (2.2.21) may be reduced in the form of ultra-spherical polynomial $P_n^{(a,a)}(x)$ by putting b = a.

On the other hand making use of a relation [88, p.277(5); see also 114, p.81] between ultraspherical polynomial $P_n^{(a,a)}(x)$ and Gegenbauer's polynomial $C_n^a(x)$

$$P_n^{(a,a)}(x) = \frac{(1+a)_n}{(1+2a)_n} C_n^{a+\frac{1}{2}}(x) , \qquad \ldots(2.2.22)$$

in (2.2.20) and (2.2.21), we can find a number of generating functions in the form of Gegenbauer's polynomial [88,p.279(15)] defined as

$$C_n^{a}(x) = \frac{(2a)_n}{n!}\ {}_2F_1 \left[\begin{matrix} -n\ ,\ 2a+n; \\ a+\frac{1}{2}\quad ; \end{matrix}\ \frac{1-x}{2} \right] . \qquad \ldots(2.2.23)$$

Again using a generating function [88, p.276(1)] for Gegenbauer's polynomial

$$\sum_{n=0}^{\infty} C_n^{a}(x)\, t^n = (1-2xt+t^2)^{-a} = (1+t^2)^{-a}\ {}_1F_0\left[\begin{matrix} a\ ; \\ -\ ; \end{matrix}\ \frac{2xt}{1+t^2}\right], \quad \ldots(2.2.24)$$

we can easily get

$$\sum_{n=0}^{\infty} t^n\, C_{2n}^{a}(x) = (1+t)^{-a}\ {}_2F_1\left[\begin{matrix} \frac{a}{2},\ \frac{a+1}{2}\ ; \\ \frac{1}{2}\ ; \end{matrix}\ \frac{4x^2t}{(1+t)^2}\right] \quad \ldots(2.2.25)$$

and

$$\sum_{n=0}^{\infty} t^n\, C_{2n+1}^{a}(x) = (1+t)^{-a-1}\, 2ax\ {}_2F_1\left[\begin{matrix} \frac{a+1}{2},\ \frac{a+2}{2}\ ; \\ \frac{3}{2}\ ; \end{matrix}\ \frac{4x^2t}{(1+t)^2}\right]. \quad \ldots(2.2.26)$$

The generating functions (2.2.24) to (2.2.26) may be reduced in the form of Legendre's polynomial of first kind $P_n(x)$ [88, p.166(2)] by putting $a = \frac{1}{2}$, Chebicheff's polynomial of second kind $U_n(x)$ by putting a = 1 and using the relations [114, p. 81]

$$C_n^{\frac{1}{2}}(x) = P_n(x) = P_n^{(o,o)}(x), \quad \ldots(2.2.27)$$

$$C_n^{1}(x) = U_n(x) = \frac{(n+1)!}{(\frac{3}{2})_n}\, P_n^{(\frac{1}{2},\frac{1}{2})}(x). \quad \ldots(2.2.28)$$

Consider another polynomial of Rainville [88, p.237(25)]

$$g_n(x) = (2x)^n \; {}_3F_1\left[\begin{matrix} \frac{-n}{2}, \frac{-n+1}{2}, 1+a \; ; & \\ & \frac{-1}{x^2} \\ 1+b \; ; & \end{matrix}\right], \quad \ldots(2.2.29)$$

which reduces into Hermite's polynomial $H_n(x)$ [88,p.191],

by setting $b = a$,

$$H_n(x) = (2x)^n \; {}_2F_0\left[\begin{matrix} \frac{-n}{2}, \frac{-n+1}{2} \; ; & \\ & \frac{-1}{x^2} \\ \text{———} \; ; & \end{matrix}\right]. \quad \ldots(2.2.30)$$

A generating function [88, p.239(40)] for $g_n(x)$ is given by

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} g_n(x) = \exp(2xt) \; {}_1F_1\left[\begin{matrix} 1+a; & \\ & -t^2 \\ 1+b; & \end{matrix}\right]. \quad \ldots(2.2.31)$$

Now using (2.2.2) in exp (2xt), separating equal and imaginary parts and using Laplace transform technique, we get

$$\sum_{n=0}^{\infty} \frac{(h)_n t^n}{(2n)!} g_{2n}(x) = \Psi_1 \left[h, 1+a; 1+b, \tfrac{1}{2}; -t, x^2 t\right] \quad \ldots(2.2.32)$$

and

$$\sum_{n=0}^{\infty} \frac{(h)_n t^n}{(2n+1)!} g_{2n+1}(x) = 2x \, \Psi_1 \left[h, 1+a; 1+b, \tfrac{3}{2}; -t, x^2 t\right], \quad \ldots(2.2.33)$$

where Ψ_1 is Humbert's double hypergeometric function (1.4.4).

Similarly from generating function [88, p.169(1)] for Legendre polynomial

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} P_n(x) = e^{xt} \, {}_0F_1\left[\begin{matrix} -\!-\!-; \\ 1\ ; \end{matrix} \ \frac{t^2(x^2-1)}{4}\right], \qquad \ldots(2.2.34)$$

we get two more generating functions

$$\sum_{n=0}^{\infty} \frac{(a)_n\, t^n}{(2n)!} P_{2n}(x) = \Psi_2\left[a;\ 1,\ \tfrac{1}{2};\ \frac{t(x^2-1)}{4},\ \frac{tx^2}{4}\right] \qquad \ldots(2.2.35)$$

and

$$\sum_{n=0}^{\infty} \frac{(a)_n\, t^n\, P_{2n+1}(x)}{(2n+1)!} = x\,\Psi_2\left[a; 1, \tfrac{3}{2};\ \frac{t(x^2-1)}{4},\ \frac{tx^2}{4}\right], \qquad \ldots(2.2.36)$$

where Ψ_2 is another Humbert's function (1.4.5).

When b = a and replacing t by $\frac{t}{h}$, taking $h \to \infty$, (2.2.32) and (2.2.33) reduce to known generating functions of Sharma [94, p.133(18, 19)]

$$\sum_{n=0}^{\infty} \frac{t^n}{(2n)!} H_{2n}(x) = e^{-t} \, {}_0F_1\left[\begin{matrix} -\!-\!-; \\ \tfrac{1}{2}\ ; \end{matrix} \ x^2 t\right], \qquad \ldots(2.2.37)$$

and

$$\sum_{n=0}^{\infty} \frac{t^n}{(2n+1)!} H_{2n+1}(x) = 2x\, e^{-t} \, {}_0F_1\left[\begin{matrix} -\!-\!-; \\ \tfrac{3}{2}\ ; \end{matrix} \ tx^2\right]. \qquad \ldots(2.2.38)$$

In (2.3.3), replacing x, z and b by $\left(\frac{2x}{c}-1\right)$, $-z(1+c)$ and ℓ, respectively, taking $c \to \infty$ and using the results (2.2.10), (2.2.11), we get

$$\sum_{n=0}^{\infty} \frac{(d)_n z^n}{(e)_n} L_n^{(a-n)}(x) \; {}_2F_1\left[\begin{matrix} -m, d+n ; \\ e+n ; \end{matrix} \; y\right]$$

$$= {}_3\Phi_D^{(1)}\left[d, -a, -m; e; -z, y, -xz\right] \qquad \ldots(2.3.4)$$

$$= \Phi_D^{(3)}\left[d; -a, -m, - ; e; -z, y, -xz\right] \qquad \ldots(2.3.5)$$

$$= F_{D_1}\left[d,d,d; -a, -m, -; e,e,e; -z, y, -xz\right], \qquad \ldots(2.3.6)$$

where ${}_3\Phi_D^{(1)}$, $\Phi_D^{(3)}$ and F_{D_1} are equivalent forms of a triple series given by (1.9.1) and (1.11.1).

Now replacing in (2.3.4) or (2.3.5) or (2.3.6), e, z and y by $1+e$, $\frac{z}{d}$ and $\frac{y}{d}$, respectively and taking $d \to \infty$, we get a bilinear generating function for Laguerre's polynomials

$$\sum_{n=0}^{\infty} \frac{m! \, z^n}{(1+e)_{m+n}} L_n^{(a-n)}(x) \; L_m^{(e+n)}(y)$$

$$= \Phi_3^{(3)}\left[-a, -m; 1+e; -z, y, -xz\right], \qquad \ldots(2.3.7)$$

where $\phi_3^{(3)}$ is a function of Exton given by (1.12.5).

Setting $z = -y$ and using a transformation of Exton [36, p.91]

$$F_{D_1}\left[a, a, a; b, c, -; e, e, e; x, x, z\right]$$

$$= \phi_1\left[a; b+c; e; x, z\right], \qquad \ldots(2.3.8)$$

(2.3.6) reduces in the form

$$\sum_{n=0}^{\infty} \frac{(d)_n(-y)^n}{(e)_n} L_n^{(a-n)}(x)\ {}_2F_1\left[\begin{matrix}-m, d+n; \\ e+n\ ;\end{matrix}\ y\right]$$

$$= \phi_1\left[d, -(a+m); e; y, xy\right], \qquad \ldots(2.3.9)$$

where ϕ_1 is Humbert's function given by (1.4.1).

Replacing y by $-y$ and taking $m = 0$, (2.3.9) reduces to a known result of Khan [57, p.183 (4.6)]

$$\sum_{n=0}^{\infty} \frac{(d)_n\, y^n}{(e)_n} L_n^{(e-n)}(x) = \phi_1\left[d; -a; e; -y, -xy\right]. \qquad \ldots(2.3.10)$$

When y is replaced by $\frac{y}{d}$ and taking $d \to \infty$, (2.3.10) reduces to another known generating function of Khan [58, p.439 (3.1)]

$$\sum_{n=0}^{\infty} \frac{y^n}{(e)_n} L_n^{(a-n)}(x) = \Phi_3\left[-a; e; -y, -xy\right], \qquad \ldots(2.3.11)$$

where Φ_3 is another Humbert's function given by (1.4.3).

When y = o or z = o, (2.3.6) reduces to (2.3.10). When x = o, (2.3.6) reduces to

$$\sum_{n=0}^{\infty} \frac{(a)_n (d)_n z^n}{(e)_n\, n!}\ {}_2F_1\left[\begin{matrix} -m, d+n ; \\ e+n ; \end{matrix}\ y\right] = F_1\left[d; a, -m; e; z, y\right], \qquad \ldots(2.3.12)$$

where F_1 is Appell's function of first kind given by (1.3.1).

Replacing z by $\frac{z}{a}$ in (2.3.12) and taking $a \to \infty$, we get

$$\sum_{n=0}^{\infty} \frac{(d)_n z^n}{(e)_n\, n!}\ {}_2F_1\left[\begin{matrix} -m, d+n ; \\ e+n ; \end{matrix}\ y\right] = \Phi_1\left[d; -m; e; y, z\right], \ldots(2.3.13)$$

whereas on replacing y and z by $\frac{y}{d}$ and $\frac{z}{d}$, respectively and taking $d \to \infty$, (2.3.12) reduces to

$$\sum_{n=0}^{\infty} \frac{m!\,(a)_n\, z^n}{(1+e)_{m+n}\, n!} L_m^{(e+n)}(y) = \Phi_2\left[a, -m; 1+e; z, y\right], \ldots(2.3.14)$$

where Φ_2 is another Humbert's function given by (1.4.2).

Similarly by a process of confluence, (2.3.14) gives

$$\sum_{n=0}^{\infty} \frac{m!\, z^n}{(1+e)_{m+n}\, n!} L_m^{(e+n)}(y) = \Phi_3\left[-m;\ 1+e;\ y, z\right]. \qquad \ldots(2.3.15)$$

When $y = o$ or $m = o$ in (2.3.7), we again get (2.3.11) and when $x = o$, (2.3.7) reduces to (2.3.14). Alternatively (2.3.4), (2.3.12) and (2.3.13) can also be written in the following bilateral and linear generating functions

$$\sum_{n=0}^{\infty} \frac{z^n (1+e+d+m)_n}{(1+e+m)_n} L_n^{(a-n)}(x)\ P_m^{(e+n,d)}(y)$$

$$= \frac{(1+e)_m}{m!}\ {}_3\Phi_D^{(1)}\left[1+e+d+m, -a, -m;\ 1+e; -z, \frac{1-y}{2}, -xz\right], \ldots(2.3.16)$$

$$\sum_{n=0}^{\infty} \frac{z^n\, (a)_n\, (1+e+d+m)_n}{(1+e+m)_n\ n!} P_m^{(e+n,d)}(y)$$

$$= \frac{(1+e)_m}{m!} F_1\left[1+e+d+m;\ a, -m;\ 1+e;\ z, \frac{1-y}{2}\right] \qquad \ldots(2.3.17)$$

and

$$\sum_{n=0}^{\infty} \frac{z^n (1+e+d+m)_n}{(1+e+m)_n\ n!} P_m^{(e+n,d)}(y)$$

$$= \frac{(1+e)_m}{m!} \Phi_1\left[1+e+d+m,\ -m;\ 1+e;\ \frac{1-y}{2}, z\right]. \qquad \ldots(2.3.18)$$

CHAPTER III

SUMMATION FORMULAE INVOLVING HYPERGEOMETRIC FUNCTIONS

3.1 INTRODUCTION :

In an attempt to generalize some results of Carlitz [17] and Halim and Salam [47]; Manocha and Sharma [70,p.475(30,32,34)] gave the following results :

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{t^n x^m (-y)^{n-m} (c)_m (d)_{n-m}}{(a)_m (b)_{n-m}\, m!\,(n-m)!}\; {}_2F_1\left[\begin{matrix} -n+m,\ c+m; \\ a+m\ ; \end{matrix}\ x\right]$$

$${}_2F_1\left[\begin{matrix} -m,\ d+n-m\ ; \\ b+n-m\ ; \end{matrix}\ y\right] = \sum_{n=0}^{\infty} t^n\; {}_3F_2\left[\begin{matrix} -n,\ c,\ 1-b-n; \\ a,\ 1-d-n\ ; \end{matrix}\ x/y\right], \qquad \ldots(3.1.1)$$

$${}_3F_2\left[\begin{matrix} -n,\ c,\ b-d\ ; \\ 1+c-a-n, 1-d-n; \end{matrix}\ 1\right] = \frac{(c)_n}{(a-c)_n}\; {}_3F_2\left[\begin{matrix} -n,c,1-b-n\ ; \\ a,1-d-n\ ; \end{matrix}\ 1\right] \qquad \ldots(3.1.2)$$

and

$$\sum_{m=0}^{n} \frac{(-b-n)_m}{(1+a)_m} L_m^{(a)}(x)\, L_{n-m}^{(b)}(-x)$$

$$= \frac{x^n (1+a+b)_{2n}}{(1+a)_n (1+b)_n (1+a+b)_n}, \qquad \ldots(3.1.3)$$

respectively.

It does not appear to have been observed previously that (3.1.1) to (3.1.3) are not correct, although many workers have used them lately.

By making use of fractional derivatives, we give corrections to results (3.1.1) and (3.1.2) in section 3.2. Later in section 3.3, we apply manipulations of series to obtain summations for the product of hypergeometric polynomials which are further specialized in terms of Jacobi and Laguerre polynomials. One of the special cases of these results yields the correct form of the result (3.1.3).

In section 3.4, a wide variety of summation results appertaining to hypergeometric functions of Appell, Gauss, Humbert and Kampé de Fériet, are obtained by operational methods from our main finite summation formula involving Srivastava's triple and Kampé de Fériet's double series. A number of results of Munot [72] involving Jacobi's and Appell's polynomials follow as special cases of our general results. Certain double infinite sums involving Appell's F_2 and Jacobi's polynomials, are also deduced.

3.2 USE OF FRACTIONAL DERIVATIVES :

In our investigations, we shall use the following result which may easily be derived from the definitions of Binomial theorem [88, p.74(3)] and fractional derivative (2.3.1).

$$D_x^{c-a}\left[x^{m+c-1}(1-x)^n\right] = \frac{x^{(m+a-1)}\,\Gamma c\,(c)_m}{\Gamma a\,(a)_m}\;{}_2F_1\left[\begin{matrix}-n, c+m; \\ a+m\;;\end{matrix}\; x\right], \qquad \ldots(3.2.1)$$

where m, n are non-negative integers and $|x| < 1$.

Consider an elementary result

$$e^{x(1-y)t}\; e^{-y(1-x)t} = e^{xt}\,.\,e^{-yt}, \qquad \ldots(3.2.2)$$

which may be written in the form

$$\sum_{m,n=0}^{\infty} \frac{(-1)^n x^m (1-y)^m y^n (1-x)^n t^{m+n}}{m!\;n!} = \sum_{m,n=0}^{\infty} \frac{(-1)^n x^m y^n t^{m+n}}{m!\;n!}. \qquad \ldots(3.2.3)$$

Now multiplying both the sides of (3.2.3) by x^{c-1}. y^{d-1}, applying the operators D_x^{c-a}, D_y^{d-b} on both the sides and using the definitions (2.3.1) and (3.2.1), we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(c)_m (d)_{n-m}\, x^m (-y)^{n-m}\, t^n}{(a)_m (b)_{n-m}\, m!\;(n-m)!}\;{}_2F_1\left[\begin{matrix}-n+m, c+m; \\ a+m\;;\end{matrix}\; x\right]$$

$${}_2F_1\left[\begin{matrix}-m,\ d+n-m;\\ b+n-m\ ;\end{matrix}\ y\right] = \sum_{n=0}^{\infty} \frac{(-y)^n (d)_n t^n}{(b)_n\ n!}\ {}_3F_2\left[\begin{matrix}-n, c, 1-b-n;\\ a, 1-d-n\ ;\end{matrix}\ x/y\right],$$

...(3.2.4)

which is the correct form of the result (3.1.1). The equation (3.2.4) requires that a, b, c, d be not zero or negative integers.

We now turn to an error in equation (3.1.2). On equating the coefficients of t^n on both the sides in (3.2.4), we get

$$\sum_{m=0}^{n} \binom{n}{m} \frac{(c)_m (1-b-n)_m}{(a)_m (1-d-n)_m} \left(\frac{-x}{y}\right)^m {}_2F_1\left[\begin{matrix}-n+m,\ c+m\ ;\\ a+m\ ;\end{matrix}\ x\right]$$

$${}_2F_1\left[\begin{matrix}-m, d+n-m;\\ b+n-m\ ;\end{matrix}\ y\right] = {}_3F_2\left[\begin{matrix}-n,\ c,\ 1-b-n;\\ a,\ 1-d-n\ ;\end{matrix}\ x/y\right], \qquad ...(3.2.5)$$

which is the result of Manocha and Sharma [7], p.475(31)]- Here $\binom{n}{m}$ is binomial coefficient.

Putting $x = y = 1$ in (3.2.5) and using Gauss's summation theorem [88, p.69 (4)],

$${}_2F_1\left[\begin{matrix}-n,\ b\ ;\\ c\ ;\end{matrix}\ 1\right] = \frac{(c-b)_n}{(c)_n}\ ; \ (c-b) \neq 0, -1, -2, \ldots, \qquad ...(3.2.6)$$

we get

$$ {}_3F_2\left[\begin{matrix} -n,\ c,\ b-d; \\ 1-d-n,1-a-n+c; \end{matrix}\ 1\right] = \frac{(a)_n}{(a-c)_n}\ {}_3F_2\left[\begin{matrix} -n,\ c,\ 1-b-n; \\ a,\ 1-d-n\ ; \end{matrix}\ 1\right], \qquad \ldots(3.2.7) $$

which is the correct form of equation (3.1.2). In fact the equation (3.2.7) is a well known result of Bailey [1[illegible], p.238(2.1); see also 1, p.1[illegible](26)].

Now replacing x, y, a, b and m in (3.2.5) by $\frac{x}{c}$, $\frac{y}{d}$, 1+a, 1+b and (n-m), respectively, making $c \to \infty$, $d \to \infty$ and using the result (2.2.1[illegible]), we get

$$ \sum_{m=0}^{n} \frac{(-b-n)_{n-m}}{(1+a)_{n-m}\ m!\ (n-m)!} \left(\frac{y}{x}\right)^m {}_1F_1\left[\begin{matrix} -m\ ; \\ 1+a+n-m; \end{matrix}\ x\right] {}_1F_1\left[\begin{matrix} -n+m\ ; \\ 1+b+m; \end{matrix}\ y\right] $$

$$ = \frac{1}{n!}\left(\frac{y}{x}\right)^n\ {}_2F_1\left[\begin{matrix} -n,\ -b-n\ ; \\ 1+a\ ; \end{matrix}\ \frac{-x}{y}\right], \qquad \ldots(3.2.8) $$

which can be put in the form of

$$ \sum_{m=0}^{n} \left(\frac{-y}{x}\right)^m L_m^{(a+n-m)}(x)\ L_{n-m}^{(b+m)}(y) $$

$$ = \left(\frac{x+y}{x}\right)^n (-1)^n\ P_n^{(a,b)}\left(\frac{y-x}{y+x}\right), \qquad \ldots(3.2.9) $$

by using the definitions of hypergeometric polynomials of Laguerre (2.2.14) and Jacobi [88, p.254(2)]

$$P_n^{(a,b)}(x) = \frac{(1+a)_n}{n!}\left(\frac{1+x}{2}\right)^n {}_2F_1\left[\begin{matrix}-n, & -b-n; \\ & \\ 1+a & ;\end{matrix}\ \frac{x-1}{x+1}\right] \text{ . } \quad \ldots(3.2.10)$$

Setting $y = -x$ in (3.2.8) and using the results (3.2.6) and (2.2.14), we get an interesting result, in which finite sum of the product of two generalised Laguerre polynomials having the same argument but opposite sign, different order and degree, is equal to a constant for every value of the argument, that is

$$\sum_{m=0}^{n} L_m^{(a+n-m)}(x)\; L_{n-m}^{(b+m)}(-x) = \frac{(1+a+b+n)_n}{n!} \text{ .} \quad \ldots(3.2.11)$$

3.3 USES OF SRIVASTAVA'S IDENTITY :

In this section, some more finite summations of the product of two hypergeometric polynomials are obtained by using the method of series manipulations. We shall use the following triple finite series identity given by Srivastava [106,p.95]

$$\sum_{m=0}^{n} \sum_{r=0}^{m} \sum_{s=0}^{n-m} A(n,m,r,s)$$
$$= \sum_{s=0}^{n} \sum_{r=0}^{n-s} \sum_{m=0}^{n-r-s} A(n,m+r,r,s), \quad \ldots(3.3.1)$$

and combinatorial identity

$$\sum_{k=o}^{n} (-1)^k \binom{n}{k} = \begin{cases} 1, \text{ if } n = o \\ o, \forall\, n \in \{1,2,3,\ldots\}. \end{cases} \qquad \ldots(3.3.2)$$

Consider the series

$$T = \sum_{m=o}^{n} \frac{(-n)_m}{m!} \; {}_2F_1\left[\begin{matrix} -m, c \;; \\ a \;; \end{matrix}\; x\right] \; {}_2F_1\left[\begin{matrix} -n+m, d \;; \\ b \;; \end{matrix}\; y\right], \qquad \ldots(3.3.3)$$

which can be put in the following forms

$$T = \sum_{m=o}^{n} \sum_{r=o}^{m} \sum_{s=o}^{n-m} \frac{(-n)_m \, (-m)_r \, (c)_r \, (m-n)_s \, (d)_s \, x^r \, y^s}{(a)_r \, (b)_s \, m! \, r! \, s!}$$

$$= \sum_{m=o}^{n} \sum_{r=o}^{m} \sum_{s=o}^{n-m} \frac{(-1)^m \, (c)_r \, (d)_s \, n! \, (-x)^r \, (-y)^s}{(a)_r \, (b)_s \, r! \, s! \, (m-r)! \, (n-m-s)!}.$$

Using (3.3.1) in the above summations, we have

$$T = \sum_{s=o}^{n} \sum_{r=o}^{n-s} \sum_{m=o}^{n-r-s} \frac{(-1)^m \, (c)_r \, (d)_s \, n! \, x^r \, (-y)^s}{(a)_r \, (b)_s \, r! \, s! \, m! (n-m-r-s)!}$$

$$= \sum_{s=o}^{n} \sum_{r=o}^{n-s} \frac{n! \, (c)_r (d)_s \, x^r \, (-y)^s}{(a)_r (b)_s \, r! \, s! (n-r-s)!} \sum_{m=o}^{n-r-s} (-1)^m \binom{n-r-s}{m}$$

$$= n! \sum_{s=o}^{n} \frac{(d)_s (-y)^s}{(b)_s \; s!} \sum_{r=o}^{n-s} \frac{(c)_r \, x^r}{(a)_r \, r! (n-r-s)!} \sum_{m=o}^{n-r-s} (-1)^m \binom{n-r-s}{m}. \qquad \ldots(3.3.4)$$

Now with the help of (3.3.2), we observe that the value of the following series

$$A_r = \sum_{r=0}^{n-s} \frac{(c)_r\, x^r}{(a)_r\, r!\,(n-r-s)!} \sum_{m=0}^{n-r-s} (-1)^m \binom{n-r-s}{m} \qquad \ldots(3.3.5)$$

will be zero for $r = 0, 1, 2, \ldots, (n-s-1)$ and thus for $r = (n-s)$, we get

$$A_{n-s} = \frac{(c)_{n-s}\, x^{n-s}}{(a)_{n-s}(n-s)!} = \frac{(c)_n(-n)_s(1-a-n)_s\, x^n}{(a)_n(1-c-n)_s\; n!} \left(\frac{-1}{x}\right)^s. \qquad \ldots(3.3.6)$$

Therefore from (3.3.3), (3.3.4) with (3.3.5) and (3.3.6), we have

$$T = \sum_{m=0}^{n} \frac{(-n)_m}{m!}\; {}_2F_1\left[\begin{matrix} -m,\ c; \\ a\ ; \end{matrix}\ x\right]\; {}_2F_1\left[\begin{matrix} -n+m,\ d; \\ b\ ; \end{matrix}\ y\right]$$

$$= \frac{(c)_n\, x^n}{(a)_n}\; {}_3F_2\left[\begin{matrix} -n,\ 1-a-n,\ d; \\ b,\ 1-c-n; \end{matrix}\ y/x\right]. \qquad \ldots(3.3.7)$$

When $x = y = 1$, (3.3.7) reduces to a transformation of ${}_3F_2$ having the unit argument

$${}_3F_2\left[\begin{matrix} -n,\ a-c,\ 1-b-n; \\ a,\ 1-b-n+d\ ; \end{matrix}\ 1\right] = \frac{(b)_n (c)_n}{(b-d)_n (a)_n}\; {}_3F_2\left[\begin{matrix} -n,\ d,\ 1-a-n; \\ b,\ 1-c-n; \end{matrix}\ 1\right]. \qquad \ldots(3.3.8)$$

Now replacing x, y, a and b in (3.3.7) by $\frac{x}{c}, \frac{y}{d}$, 1+a and 1+b, respectively and taking the limits $c \to \infty$, $d \to \infty$, we get

$$\sum_{m=0}^{n} \frac{(-n)_m}{m!}\ {}_1F_1\left[\begin{matrix} -m\ ; \\ 1+a\ ; \end{matrix}\ x\right]\ {}_1F_1\left[\begin{matrix} -n+m\ ; \\ 1+b\ ; \end{matrix}\ y\right]$$

$$= \frac{x^n}{(1+a)_n}\ {}_2F_1\left[\begin{matrix} -n,\ -a-n\ ; \\ 1+b\ ; \end{matrix}\ \frac{-y}{x}\right], \qquad \ldots(3.3.9)$$

which reduces to a known result of Manocha and Sharma [70, p.475(33)]

$$\sum_{m=0}^{n} \frac{(-b-n)_m}{(1+a)_m}\ L_m^{(a)}(x)\ L_{n-m}^{(b)}(y)$$

$$= \frac{(-1)^n\ (x+y)^n}{(1+a)_n}\ P_n^{(a,b)}\left(\frac{y-x}{y+x}\right), \qquad \ldots(3.3.10)$$

by using the definitions of hypergeometric polynomials of Laguerre (2.2.14) and Jacobi [86, p.255(8)]

$$P_n^{(a,b)}(x) = \frac{(1+b)_n}{n!}\left(\frac{x-1}{2}\right)^n\ {}_2F_1\left[\begin{matrix} -n,-a-n; \\ 1+b\ ; \end{matrix}\ \frac{x+1}{x-1}\right]. \quad \ldots(3.3.11)$$

Setting $y = -x$ in (3.3.9), we get after a little simplification

$$\sum_{m=0}^{n} \frac{(-b-n)_m}{(1+a)_m} L_m^{(a)}(x)\, L_{n-m}^{(b)}(-x)$$

$$= \frac{(1+a+b)_{2n}\, x^n}{(1+a)_n\, (1+a+b)_n\, n!}, \qquad \ldots(3.3.12)$$

which is the correct form of equation (3.1.3).

Furthermore, Manocha and Sharma [70] have stated that equations (3.3.10) and (3.3.12) are confluent cases of the result [70, p.475(31)], that is, the equation (3.2.5). It may be observed that these equations are not confluent cases of the result [70, p.475(31)]. In fact the confluent cases of the result [70, p.475(31)] are the equations (3.2.9) and (3.2.11).

Similarly by making use of Srivastava's identity (3.3.1), we obtain one more finite summation formula

$$\sum_{m=0}^{n} \frac{(-n)_m}{m!} F_3\left[-m, m-n; a, b; c; x, y\right]$$

$$= \frac{x^n (a)_n}{(c)_n}\; {}_2F_1\left[\begin{matrix} -n, b\,; \\ 1-a-n\,; \end{matrix}\; \frac{-y}{x}\right], \qquad \ldots.(3.3.13)$$

where F_3 is Appell's polynomial of third kind.

3.4 USE OF [illegible] INTEGRAL :

For Jacobi's and Appell's polynomial of second kind, Munot [72, p.694 (3.4, 3.1, 3.2)] established the following three results :

$$\sum_{m,n=0}^{\infty} \frac{t^{m+n}(b)_n}{n!} P_m^{(a,d+n)}(x)$$

$$= 2^{a+d} R^{-1} (1+t+R)^{b-d} (1-t+R)^{-a-b}, \qquad \ldots(3.4.1)$$

$$\sum_{m=0}^{n} \frac{(a)_m}{m!} P_{n-m}^{(b,d+m)}(x) = P_n^{(a+b,d-a)}(x) \qquad \ldots(3.4.2)$$

and

$$\sum_{m=0}^{n} \frac{(1+a)_{n-m}(b)_m}{m!(n-m)!} F_2\left[A; m-n, B; 1+a, C; x, z\right]$$

$$= \frac{(1+a+b)_n}{n!} F_2\left[A, -n, B; 1+a+b, C; x, z\right], \qquad \ldots(3.4.3)$$

where $\quad R = (1-2xt+t^2)^{1/2}. \qquad \ldots(3.4.4)$

In this section, we shall prove the following more general relations

$$\sum_{m,n=0}^{\infty} \frac{t^{m+n}(1+a)_m(1+b)_n}{m!\,n!} F_2\left[2+a+b+d+m+n; -m,-n; 1+a, 1+b; \frac{1-x-py}{2}, \frac{(p-1)y}{2}\right]$$

$$= 2^{1+a+b+d} H^{-1} (1+t+H)^{-d} (1-t+H)^{-1-a-b}, \qquad \ldots(3.4.5)$$

where

$$H = (1-2xt - 2yt + t^2)^{1/2} \qquad \ldots(3.4.6)$$

and p is any number, and

$$\sum_{m,n=0}^{\infty} t^{m+n} P_m^{(a,e-m)}(x) P_n^{(b,\, m+d-e)}(x)$$

$$= 2^{1+a+b+d} S^{-1} (1+t+S)^{-d} (1-t+S)^{-1-a-b}, \qquad \ldots(3.4.7)$$

where S is given by the equation (3.4.4).

These results follow directly by operational methods from a finite summation formula obtained in the form

$$F^{1:2;1}_{1:1;0}\left[\begin{matrix} A : B, -n; & C-B; \\ C : 2+a+b; & -\!-; \end{matrix} \; \frac{x+y}{1+z}, \frac{z}{1+z}\right]$$

$$= \frac{n!}{(2+a+b)_n} \sum_{m=0}^{n} \frac{(1+a)_m (1+b)_{n-m}}{m!\,(n-m)!}$$

$$. F^{(3)}\left[\begin{matrix} A::B; -;- : -m; m-n; C-B; \\ C::-; -; -:1+a; 1+b; -\!-; \end{matrix} \; \frac{x}{1+z}, \frac{y}{1+z}, \frac{z}{1+z}\right], \qquad \ldots(3.4.8)$$

where $F^{1:2;1}_{1:1;0}$ and $F^{(3)}$ are given by means of the equations (1.6.1) and (1.10.2), respectively.

To prove (3.4.8), we shall make use of the following results:

A well known result [43, p.27(3.17)] is given in the form

$$L_n^{(a+b+1)}(x+y) = \sum_{m=0}^{n} L_m^{(a)}(x)\, L_{n-m}^{(b)}(y). \qquad \ldots(3.4.9)$$

A generalisation of the Laplace type single integrals of Slater [96, p.54; see also 104, p.680(1.5)], Erdélyi, A., et al. [35, p.216(16) and 73, p.189(17, 105)], [35, p.215(11)], [35, p.174(29) and 34, p.191 (33)], Kulshreshtha [59, pp.226(2.2), 227(2.4); see also 80, p.286(1.6)] and Goldstein [45, p.114(54)], was obtained by Pathan [82, p.785(2.2)] in the form

$$I_2 = \int_0^{\infty} t^{g} \exp\left\{-\left(s+\frac{p}{2}\right)t\right\} W_{k,h}(pt)\, L_{n_1}^{(2m_1)}(xt)\, L_{n_2}^{(2m_2)}(yt)\,dt$$

$$= \frac{(2m_1+1)_{n_1}(2m_2+1)_{n_2}\,\Gamma(g+h+\frac{3}{2})\,\Gamma(g-h+\frac{3}{2})\,p^{(h+\frac{1}{2})}}{(n_1)!\,(n_2)!\,\Gamma(g-k+2)\,(s+p)^{g+h+3/2}}\,.$$

$$F^{(3)}\left[\begin{array}{l} g+h+\frac{3}{2} :: g-h+\frac{3}{2}; -; -: -n_1; -n_2; h-k+\frac{1}{2}; \\ g-k+2 :: \text{——}; -; -: 1+2m_1; 1+2m_2; \text{——}; \end{array} \frac{x}{s+p}, \frac{y}{s+p}, \frac{s}{s+p}\right]. \quad \ldots(3.4.10)$$

provided that $Re(g\pm h+\frac{1}{2}) > 0$ and $Re(s+p) > 0$. Here $W_{k,h}(pt)$ is Whittaker's confluent hypergeometric function [64, p.352; see also 35, p.264 (5)] given by

$$W_{k,h}(z) = e^{-\frac{z}{2}}\,(z)^{k}\;{}_2F_0\left[\begin{matrix}\frac{1}{2}-k+h,\ \frac{1}{2}-k-h\ ; \\ \text{———————} ;\end{matrix}\ \frac{-1}{z}\right], \quad \ldots(3.4.11)$$

if $|\arg(z)| < \pi$.

Now in (3.4.9), replacing x and y by xt and yt, respectively, multiplying both the sides by $t^{g}\exp\left\{-(z+\frac{p}{2})\,t\right\}\ W_{k,h}(pt)$, integrating term by term, with respect to t from 0 to ∞, interchanging the orders of summation and integration, using the Pathan's integral (3.4.10) and making suitable adjustment of parameters, we get our main finite summation formula (3.4.8).

When a transformation formula [81, p.372 (1.2)] of Pathan given in the form

$$F^{(3)}\left[\begin{matrix}a::\ b;\ -;\ -:\ d\ ;\ f;\ \ c-b\ ; \\ c::\ -;\ -;\ -:\ e\ ;\ g;\ \text{———}\ ;\end{matrix}\ \frac{x}{1+z},\ \frac{y}{1+z},\ \frac{z}{1+z}\right]$$

$$= (1+z)^{c-b}\ F^{(3)}\left[\begin{matrix}-::\ a,b;\ -;\ -:\ d;\ f;\ c-a,c-b; \\ c::\ \text{——};\ -;\ -:\ e;\ g;\ \text{————};\end{matrix}\ \frac{x}{1+z},\frac{y}{1+z},\ -z\right], \quad \ldots(3.4.12)$$

is used for $F^{(3)}$ in (3.4.8), we get the following alternative form of (3.4.8),

$$F^{1:2;1}_{1:1;o}\left[\begin{matrix} A : B, -n; C-B ; \\ C : 2+a+b; --- ; \end{matrix} \frac{x+y}{1+z}, \frac{z}{1+z}\right]$$

$$= \frac{n!\,(1+z)^{C-B}}{(2+a+b)_n} \sum_{m=o}^{n} \frac{(1+a)_m\,(1+b)_{n-m}}{m!\,(n-m)!} .$$

$$F^{(3)}\left[\begin{matrix} --- :: A, B; -; -: -m; m-n; C-A, C-B; \\ C :: ---; -; -:1+a; 1+b; -------; \end{matrix} \frac{x}{1+z}, \frac{y}{1+z}, -z\right] . \qquad \ldots(3.4.13)$$

Now we apply (3.4.8) and (3.4.13) in a number of different directions to obtain the various finite and infinite sums.

When $z = o$ and $C = B$, (3.4.8) reduces in the form

$$\sum_{m=o}^{n} \frac{(1+a)_m\,(1+b)_{n-m}}{m!\,(n-m)!} F_2\left[A; -m, m-n; 1+a, 1+b; x, y\right]$$

$$= \frac{(2+a+b)_n}{n!}\ {}_2F_1\left[\begin{matrix} -n, A ; \\ 2+a+b ; \end{matrix} x+y\right] . \qquad \ldots(3.4.14)$$

Now using the transformations given in the book of Exton [41, pp.215 (6.9.2), 216(6.9.6); see also 21, p.552(3.2)]

$$ {}_2F_1\left[\begin{matrix}-n,\ b;\\ c\ ;\end{matrix}\ 1-x\right] = \frac{(c-b)_n}{(c)_n}\ {}_2F_1\left[\begin{matrix}-n,\ b\ ;\\ b-c-n+1;\end{matrix}\ x\right] \qquad \ldots(3.4.15) $$

and

$$ {}_2F_1\left[\begin{matrix}-n\ ,\ b;\\ c\ \ ;\end{matrix}\ x\right] = \frac{(b)_n(-x)^n}{(c)_n}\ {}_2F_1\left[\begin{matrix}-n,\ -n-c+1\ ;\\ -n-b+1\ \ ;\end{matrix}\ \frac{1}{x}\right], \qquad \ldots(3.4.16) $$

for Gauss's ordinary hypergeometric polynomial ${}_2F_1$ in (3.4.14), we get the following alternative forms of (3.4.14),

$$ \sum_{m=0}^{n} \frac{(1+a)_m(1+b)_{n-m}}{m!\ (n-m)!}\ F_2\left[A;\ -m, m-n; 1+a,\ 1+b; x, y\right] $$

$$ = \frac{(A)_n(1-x-y)^n}{n!}\ {}_2F_1\left[\begin{matrix}-n,\ a+b+2-A;\\ 1-A-n\ \ ;\end{matrix}\ \frac{1}{1-x-y}\right] \qquad \ldots(3.4.17) $$

$$ = \frac{(2+a+b-A)_n}{n!}\ {}_2F_1\left[\begin{matrix}-n,\ A\ \ ;\\ A-a-b-n-1;\end{matrix}\ 1-x-y\right] \qquad \ldots(3.4.18) $$

$$ = \frac{(-1)^n\ (A)_n\ (x+y)^n}{n!}\ {}_2F_1\left[\begin{matrix}-n,\ -(n+1+a+b);\\ 1-n-A\ \ ;\end{matrix}\ \frac{1}{x+y}\right] \qquad \ldots(3.4.19) $$

$$ = \frac{(2+a+b-A)_n(x+y)^n}{n!}\ {}_2F_1\left[\begin{matrix}-n,-(n+1+a+b);\\ -(n+1+a+b-A);\end{matrix}\ \frac{x+y-1}{x+y}\right] \qquad \ldots(3.4.20) $$

In (3.4.14), replacing A, x and y by $2+a+b+d+n$, $\frac{1-x-py}{2}$ and $\frac{(p-1)y}{2}$, respectively and multiplying both the sides by t^n, taking the sum from $n = o$ to ∞, using the linear generating function [34, p.172 (29); see also 88, p.271 (6)] for Jacobi's polynomial

$$\sum_{n=o}^{\infty} P_n^{(a,b)}(x)\, t^n = 2^{a+b}\, E^{-1}\, (1+t+E)^{-b}\, (1-t+E)^{-a} \quad , \qquad \ldots(3.4.21)$$

where E is given by means of the equation (3.4.4) and applying series manipulation formula [88, p.57 (2)]

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} B(k,n) = \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} B(k, n+k), \qquad \ldots(3.4.22)$$

we have (3.4.5).

Setting $p = 1$ or $\frac{1-x}{y}$ in (3.4.5), we get

$$\sum_{m,n=o}^{\infty} \frac{t^{m+n}\,(1+b)_n}{n!}\, P_m^{(a,1+b+d+n)}(x+y)$$

$$= 2^{1+a+b+d}\, H^{-1}\, (1+t+H)^{-d}\, (1-t+H)^{-1-a-b} \quad , \qquad \ldots(3.4.23)$$

where H is given by the equation (3.4.6).

Replacing d by $d-b$, b by $b-1$ and $y = o$, we get (3.4.1) from (3.4.23).

Now in (3.4.14), replacing x and y by xt and $x(1-t)$, respectively, multiplying both the sides by $t^{D-1}\,(1-t)^{A-D-1}$, integrating with respect to t between the limits $t = o$ to 1 and using the integral [88, pp. 18(1), 19(4)] for Beta function

$$I_3 = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)};\ Re(m) > o, Re(n) > o, \qquad \ldots(3.4.24)$$

we get a finite sum for the product of two Gauss's hypergeometric polynomials,

$$ {}_2F_1\left[\begin{matrix}-n, A ; \\ \\ 2+a+b ;\end{matrix}\ x\right] = \frac{n!}{(2+a+b)_n} \sum_{m=o}^{n} \frac{(1+a)_m\,(1+b)_{n-m}}{m!\ (n-m)!} \,.$$

$$ {}_2F_1\left[\begin{matrix}-m, D ; \\ \\ 1+a ;\end{matrix}\ x\right] {}_2F_1\left[\begin{matrix}m-n, A-D ; \\ \\ 1+b ;\end{matrix}\ x\right] . \qquad \ldots(3.4.25)$$

In (3.4.25), replacing A, D and x by $(2+a+b+n+d)$, $(1+a+\ell)$ and $\frac{1-x}{2}$, respectively, multiplying both the sides by t^n, taking the summation from $n = o$ to ∞ and using the relations (3.4.21) and (3.4.22), we get (3.4.7).

When $e = o$, (3.4.7) reduces in the form

$$\sum_{m,n=o}^{\infty} \frac{t^{m+n}\,(1+a)_m}{m!} \left(\frac{1+x}{2}\right)^m P_n^{(b,\,m+d)}(x)$$

$$= 2^{1+a+b+d}\, E^{-1}\,(1+t+E)^{-d}\,(1-t+E)^{-1-a-b} \quad . \qquad \ldots(3.4.26)$$

Again in (3.4.14), putting $x = o$, replacing a, y and A by $(a-1)$, $\frac{1-x}{2}$ and $(1+b+d+n)$, respectively, we get (3.4.2).

When $D = A$ in (3.4.25) then adopting the method of the derivation of (3.4.7), we again have the result (3.4.23).

Setting $y = 1$ in (3.4.14) and using the Gauss's summation theorem (3.2.6), we get

$$\sum_{m=o}^{n} \frac{(-n)_m (1+a)_m}{m!\,(A-b-n)_m}\ {}_3F_2\left[\begin{matrix} -m,\ A,\ A-b\ ; \\ 1+a,\ A-b-n+m\ ; \end{matrix}\ x\right]$$

$$= \frac{(2+a+b)_n}{(1+b-A)_n}\ {}_2F_1\left[\begin{matrix} -n\ ,\ A\ ; \\ 2+a+b\ ; \end{matrix}\ 1+x\right] \quad . \qquad \ldots(3.4.27)$$

Setting $x = o$, $B = 2+a+b$ in (3.4.8), replacing y and z by $\frac{y}{A}$ and $\frac{z}{A}$, respectively, taking $A \to \infty$ and using the result (2.2.10), we have

$$\Phi_2\left[-n,\ C-2-a-b;\ C;\ y,\ z\right] = \frac{n!}{(2+a+b)_n} \sum_{m=0}^{n} \frac{(1+a)_m}{m!} \cdot$$

$$\frac{(1+b)_{n-m}}{(n-m)!}\ F^{0:2;1}_{1:1;0}\left[\begin{matrix} \text{---} : m-n,\ 2+a+b;\ C-2-a-b; \\ C : \quad 1+b\ ;\ \text{------}; \end{matrix}\ y, z\right],$$

...(3.4.28)

where Φ_2 is Humbert's function given by the equation (1.4.2). Again, on replacing y and z by $\frac{y}{\lambda}$ and $\frac{z}{\lambda}$, respectively in (3.4.8), taking $A \to \infty$, $x = o$ and $B = 1 + b$, we find

$$F^{0:2;1}_{1:1;0}\left[\begin{matrix} \text{---} : 1+b,\ -n\ ;\ C-b-1\ ; \\ C : 2+a+b\ ;\ \text{------}; \end{matrix}\ y,\ z\right] = \frac{n!}{(2+a+b)_n} \cdot$$

$$\sum_{m=0}^{n} \frac{(1+a)_m (1+b)_{n-m}}{m!\ (n-m)!}\ \Phi_2\left[m-n,\ C-1-b,\ C;\ y,\ z\right].$$

...(3.4.29)

When $x = o$, $A = 1+b$ and $B = 2 + a + b$, (3.4.13) reduces in the form

$$F_1\left[1+b;-n, C-2-a-b;\ C;y,\ \frac{z}{1+z}\right] = \frac{n!\,(1+z)^{C-2-a-b}}{(2+a+b)_n} \cdot$$

$$\sum_{m=0}^{n} \frac{(1+a)_m (1+b)_{n-m}}{m!\ (n-m)!}\ F_3\left[m-n, C-2-a-b; 2+a+b, C-1-b; C; y, -z\right].$$

...(3.4.30)

Setting $y = o$, $C-B = D$ in (3.4.8) and making suitable adjustment of variables, we have

$$\sum_{m=o}^{n} \frac{(1+a)_m (1+b)_{n-m}}{m!\,(n-m)!} F^{1:2;1}_{1:1;0}\left[\begin{matrix} A : -m, B; D ; \\ C : 1+a; \, - ; \end{matrix} \; x, z\right]$$

$$= \frac{(2+a+b)_n}{n!} F^{1:2;1}_{1:1;o}\left[\begin{matrix} A : -n , B; D ; \\ C : 2+a+b; \, - ; \end{matrix} \; x, z\right]. \qquad \ldots(3.4.31)$$

Now replacing in (3.4.31) x, z and m by $\frac{xC}{B}$, zC and $n-m$, respectively, tending B and C to ∞, we get

$$\sum_{m=o}^{n} \frac{(1+b)_m (1+a)_{n-m}}{m!\,(n-m)!} F^{1:1;1}_{0:1;o}\left[\begin{matrix} A : m-n ; \, ; \\ - : 1+a ; - ; \end{matrix} \; x, z\right]$$

$$= \frac{(2+a+b)_n}{n!} F^{1:1;1}_{0:1;o}\left[\begin{matrix} A : - n ; D ; \\ - : 2+a+b; \, - ; \end{matrix} \; x, z\right]. \ldots(3.4.32)$$

Again, replacing z by $\frac{z}{t}$, b by $(b-1)$, multiplying both the sides of (3.4.32) by t^{-G}, taking its inverse Laplace transform with respect to t and using the integral [29, p.723 (last line); see also 105, p.40(2.6)]

$$I_4 = \int_{C-i\infty}^{C+i\infty} e^t \, t^{-G} \, dt = \frac{2\pi i}{\Gamma G} \; ; \; Re(G) > o, \qquad \ldots(3.4.33)$$

we get a known result of Munot (3.4.3).

It will be assumed in (3.4.33) that C is real and positive and contour of integration is a straight line parallel to the imaginary axis, in the positive half-plane with indentations if necessary, to avoid the poles of integrand.

When s is replaced by $\frac{s}{D}$ and $D \to \infty$, (3.4.3) reduces in the form

$$\sum_{m=0}^{n} \frac{(1+a)_{n-m}(b)_m}{m!\,(n-m)!} \Psi'_1 \left[A; m-n; 1+a, G; x, s \right]$$

$$= \frac{(1+a+b)_n}{n!} \Psi'_1 \left[A; -n; 1+a+b, G; x, s \right], \qquad \ldots(3.4.34)$$

where Ψ'_1 is Humbert's function given by the equation (1.4.4).

It is to be noted that the result (3.4.2) is easily deducible from (3.4.3) by taking s = o or D = o with suitable adjustment of parameter and variable . However , Munot [72] has obtained (3.4.2) by using the operational Calculus to (3.4.3).

When s = o, (3.4.8) reduces to

$$\sum_{m=0}^{n} \frac{(1+a)_m (1+b)_{n-m}}{m!\ (n-m)!}\ F^{2:1;1}_{1:1;1}\left[\begin{matrix} A, B : -m; m-n; \\ C : 1+a; 1+b; \end{matrix}\ x, y\right]$$

$$= \frac{(2+a+b)_n}{n!}\ {}_3F_2\left[\begin{matrix} -n, A, B ; \\ C, a+2+b; \end{matrix}\ x+y\right] . \qquad \ldots(3.4.35)$$

The form of (3.4.35) suggests the existence of the more general summation formula ,

$$\sum_{m=0}^{n} \frac{(1+a)_m (1+b)_{n-m}}{m!\ (n-m)!}\ F^{d:1;1}_{g:1;1}\left[\begin{matrix} (D_d) : -m; m-n; \\ (G_g) : 1+a; 1+b ; \end{matrix}\ x, y\right]$$

$$= \frac{(2+a+b)_n}{n!}\ {}_{d+1}F_{g+1}\left[\begin{matrix} -n, (D_d) ; \\ 2+a+b, (G_g) ; \end{matrix}\ x + y\right] , \qquad \ldots(3.4.36)$$

where (D_d) etc. are abbreviated for the array of d parameters $D_1, D_2, \ldots, D_d$.

A proof of (3.4.36), by the principle of multidimensional mathematical induction requires the Laplace and the inverse Laplace transform techniques. Indeed it holds for $d-2=g-1=0$ by virtue of (3.4.35). Let us, therefore assume that it remains true for some values of d and g. Replace x by xt and y by yt in (3.4.36), multiply both the sides by $t^{D_{d+1}-1}$, take their Laplace transform with respect to t, we find that d is replaced by $d+1$, thus completing the induction with respect to d. To effect the induction with respect to g, replace x by $\frac{x}{t}$, y by $\frac{y}{t}$ in (3.4.36), multiply both the sides by $t^{-G_{g+1}}$, take inverse Laplace transform with respect to t and using the integral (3.4.33), we find that g is replaced by $g+1$. Thus this process completes the induction on numerator and denominator parameters.

CHAPTER IV

TRANSFORMATION AND REDUCTION FORMULAE OF LAURICELLA'S FUNCTIONS

4.1 INTRODUCTION :

In a paper of Exton [40, pp.66(3.2),67(3.5,3.6)], three reduction formulae for Lauricella's triple hypergeometric function of second order F_4 (that is F_E in Saran's revised notation) are given in the form

$$F_E\left[a,\ a,\ a;\ b_1,\ b_2,\ b_2;\ c_1,\ c_2,\ c_3;\ x,\ y,\ y\right]$$

$$= F^{1:3;3}_{0:3;3}\left[\begin{matrix} a : b_1,\ A,\ B\ ;\ b_2,\ (c_2+c_3-1)/2,\ (c_2+c_3)/2; \\ -\!-: b_2,\ A,\ B\ ;\ c_2,\quad c_3,\ c_2+c_3-1\quad ; \end{matrix}\ x, 4y\right], \qquad \ldots(4.1.1)$$

$$F_E\left[a,\ a,\ a;\ b_1,\ b_2,\ b_2;\ 2b_1,\ b_2,\ b_2;\ -4y,\ y,\ y\right]$$

$$= {}_4F_3\left[\begin{matrix} a/2,\ (a+1)/2,\ \frac{b_1+b_2}{2}-\frac{1}{4},\ \frac{b_1+b_2}{2}-\frac{3}{4}\ ; \\ b_1+\frac{1}{2},\ b_2+\frac{1}{4},\ b_1+b_2-\frac{1}{2}\quad ; \end{matrix}\ 16y^2\right] \qquad \ldots(4.1.2)$$

and

$$F_E\left[a, a, a;\ b_1, b_2;\ 2b_1,\ b_2,\ b_2;\ 4y\ ,\ y,\ y\right]$$

$$=(1-y)^{-a}\ {}_4F_3\left[\begin{matrix} (2b_1+2b_2+1)/4,\ (2b_1+2b_2-1)/4,\ a/2,\ (a+1)/2; \\ b_1+\frac{1}{2},\ b_2\ ,\ b_1+b_2-\frac{1}{2}\quad ; \end{matrix}\ 16y^2(1-y)^2\right], \qquad \ldots(4.1.3)$$

where $F^{1:3;3}_{0:3;3}$ and ${}_4F_3$ are Kampé de Fériet's double hypergeometric function and single hypergeometric function, respectively.

In a book of Exton [41, pp.134 (4.7.8, 4.7.11, 4.7.12)] the following reduction formulae of F_E are given

$$F_E\left[a,a,a;\, b_1,\, b_2,\, b_2;\, c_1,\, c_2,\, c_3;\, x,\, y,\, y\right]$$

$$= F^{1:3;3}_{0:3;3}\left[\begin{matrix} a : b_1,\, A,\, B; b_2,\, (c_2+c_3-1)/2,\, (c_2+c_3)/2; \\ --: b_2,\, A,\, B;\ c_1,\ c_3,\ c_2+c_3-1; \end{matrix}\ x, 4y\right], \qquad \ldots(4.1.4)$$

$$F_E\left[a,a,a;\, b_1,\, b_2,\, b_2;\, 2b_1,\, b_2,\, b_2;\, -4y,\, y,\, y\right]$$

$$= {}_4F_3\left[\begin{matrix} a/2,\, (a+1)/2,\, \dfrac{b_1+b_2}{2}-\dfrac{1}{4},\, \dfrac{b_1+b_2}{2}-\dfrac{3}{4}; \\ b_1+\dfrac{1}{2},\, b_2+\dfrac{1}{4},\ b_1+b_2-\dfrac{1}{2}\ ; \end{matrix}\ 16y^2\right] \qquad \ldots(4.1.5)$$

and

$$(1-y)^a F_E\left[a,a,a;\, b_1, b_2, b_2;\, 2b_1,\, b_2,\, b_2;\, 4y,\, y,\, y\right]$$

$$= {}_4F_3\left[\begin{matrix} (2b_1+2b_2+1)/4,\, (2b_1+2b_2-1)/4;\, a/2,\, (a+1)/2; \\ b_1+\dfrac{1}{2},\ b_2,\ b_1+b_2-\dfrac{1}{2}\ ; \end{matrix}\ 16y^2(1-y2)\right]. \qquad \ldots(4.1.6)$$

Another reduction formula of Lauricella's triple hypergeometric function of second order F_{14}(that is F_F in Saran's revised notation) given in the work of Exton [41 , p.116(4.1.17)] is

$$F_F\left[a, a, a; b_1, b_2, b_1; a, a, a; x, y, z\right]$$

$$= (1-x-z)^{-b_1} (1-y)^{-b_2} H_3\left[b_1, b_2; a; \frac{xz}{(1-x-z)^2}, \frac{xy}{(1-y)}\right], \qquad \ldots(4.1.7)$$

where H_3 is Horn's double hypergeometric function of second order given by the equation (1.5.3).

The section 4.2 is motivated by Exton's work, which contains among other results, the above seven erroneous reduction formulae for Lauricella's triple hypergeometric functions F_E and F_F of second order. Our proofs will not be along the lines followed by Exton, who used the Laplace integral representation and some other known results. Instead we shall deduce them by series manipulation techniques.

Some more linear and quadratic reductions and transformations of F_E into Kampé de Fériet's, Horn's and Srivastava's functions, are given in section 4.3.

4.2 CORRECTION TO EXTON'S REDUCTION FORMULAE :

We write F_E (1.7.4) in the form

$$F_E\left[a, a, a; b, c, c; d, e, f; x, y, z\right]$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m (b)_m x^m}{(d)_m \, m!} F_4\left[a+m; c; e, f; y, z\right], \quad \ldots (4.2.1)$$

where F_4 is Appell's function of fourth kind given by the equation (1.3.4).

In (4.2.1), putting $z = y$ and using a reduction formula of Burchnall [13, p.101(37), see also 16, p. 439 (3.2) and 46, p. 81]

$$F_4\left[a;b;c,d;x,x\right] = {}_4F_3\left[\begin{matrix} a, b, (c+d-1)/2, (c+d)/2; \\ c, d, \quad c+d-1 \quad ; \end{matrix} \; 4x\right], \quad \ldots (4.2.2)$$

then expressing ${}_4F_3$ into power series form (1.2.1) and using the definition (1.6.1) of Kampé de Fériet's double hypergeometric function, we get the following corrected form of the reduction formulae (4.1.1) and (4.1.4) of Exton,

$$F_E\left[a, a, a; b, c, c; d, e, f; x, y, y\right]$$

$$= F^{1:1;3}_{0:1;3}\left[\begin{matrix} a: b; c, (e+f-1)/2, (e+f)/2; \\ -: d; e, f, e+f-1 \quad ; \end{matrix} \; x, 4y\right]. \qquad \ldots(4.2.3)$$

The following special cases of (4.2.3) are worthy of note :

In (4.2.3), putting $x = -4y$, $d = 2b$, $f = e = c$, we get

$$F_E\left[a, a, a; b, c, c; 2b, c, c; -4y, y, y\right]$$

$$= F_2\left[a; c-\tfrac{1}{2}, b; 2c-1, 2b; 4y, -4y\right], \qquad \ldots(4.2.4)$$

where F_2 is Appell's function of second kind given by the equation (1.3.2). Now using a transformation of Bailey [9, p.11(3.1); see also 10, p.239(4.7)]

$$F_2\left[a; d, e; 2d, 2e; 2x, 2y\right]$$

$$= (1-x-y)^{-a} F_4\left[\frac{a}{2}, \frac{a+1}{2}; d+\tfrac{1}{2}, e+\tfrac{1}{2}; \left(\frac{x}{1-x-y}\right)^2, \left(\frac{y}{1-x-y}\right)^2\right], \qquad \ldots(4.2.5)$$

in (4.2.4), we get

$$F_E\left[a,\ a,\ a;\ b,\ c,\ c;\ 2b,\ c,\ c;\ -4y,\ y,\ y\right]$$

$$= F_4\left[\ \frac{a}{2};\ \frac{a+1}{2};\ c\ ,\ b+\frac{1}{2};\ 4y^2\ ,\ \ 4y^2\right]. \qquad \ldots(4.2.6)$$

Applying the result (4.2.2) of Burchnall, in (4.2.6), we get

$$F_E\left[a,\ a,\ a;\ b,\ c,\ c;\ 2b,\ c,\ c;\ -4y,\ y,\ y\right]$$

$$= {}_4F_3\left[\begin{matrix} a/2,\ (a+1)/2,\ (2b+2c-1)/4,\ (2b+2c+1)/4; \\ \\ c,\ \ (2b+1)/2,\ \ (2b+2c-1)/2 \end{matrix}\ \ \begin{matrix} \\ 16y^2 \\ ; \end{matrix}\right], \qquad \ldots(4.2.7)$$

which is the correct form of Exton's reduction formulae (4.1.2) and (4.1.5).

Again, putting $x = 4y,\ d = 2b,\ f = e = c$ in (4.2.3) and using the result (4.2.5) of Bailey, we get

$$F_E\left[a,\ a,\ a;\ b,\ c,\ c;\ 2b,\ c,\ c;\ 4y,\ y,\ y\right]$$

$$= F_2\left[a;\ b,\ c-\frac{1}{2};\ 2b,\ 2c-1;\ \ 4y,\ 4y\right], \qquad \ldots(4.2.8)$$

$$= (1-4y)^{-a}\ F_4\left[\frac{a}{2};\frac{a+1}{2};\ \frac{2b+1}{2},\ c;\ \left(\frac{2y}{1-4y}\right)^2,\left(\frac{2y}{1-4y}\right)^2\ \right], \ldots(4.2.9)$$

which on using the result (4.2.2) of Burchnall, reduces to another correct form of the results (4.1.3) and (4.1.6) of Exton,

$$F_E\left[a, a, a; b, c, c; 2b, c, c; 4y, y, y\right]$$

$$=(1-4y)^{-a}\ {}_4F_3\left[\begin{matrix} a/2, (a+1)/2, (2b+2c+1)/4, (2b+2c-1)/4; \\ c,\ (2b+1)/2,\ (2b+2c-1)/2 \qquad ; \end{matrix}\ 16y^2/(1-4y)^2\right].$$

...(4.2.10)

Next, we consider F_P (1.7.5) in the form

$$F_P\left[a, a, a; b, c, b; a, a, a; x, y, z\right]$$

$$= \sum_{m,p=0}^{\infty} \frac{(a)_{m+p}\ (b)_{m+p}\ x^m\ z^p}{(a)_m\ \ (a)_p\ \ m!\ \ \ p!}\ {}_2F_1\left[\begin{matrix} a+m+p,\ c; \\ a+p\ ; \end{matrix}\ y\right], \quad ...(4.2.11)$$

where ${}_2F_1$ is Gauss's ordinary hypergeometric function given by (1.1.1).

Now using Euler's first linear transformation [62, p.247 (9.5.1)]

$${}_2F_1\left[\begin{matrix} a,\ b\ ; \\ c\ ; \end{matrix}\ x\right] = (1-x)^{-a}\ {}_2F_1\left[\begin{matrix} a, c-b; \\ c\ ; \end{matrix}\ \frac{x}{x-1}\right];\ |\arg(1-x)| < \pi$$

...(4.2.12)

in (4.2.11), we get

$$F_P\left[a, a, a; b, c, b; a, a, a; x, y, z\right]$$

$$= (1-y)^{-c} \sum_{m,p=0}^{\infty} \frac{(a)_{m+p}\,(b)_{m+p}\,x^m z^p}{(a)_m\,(a)_p\,m!\,p!}\;{}_2F_1\left[\begin{matrix} -m,\ c\ ; \\ a+p\ ; \end{matrix}\ \frac{y}{y-1}\right]$$

$$= (1-y)^{-c} \sum_{m,p=0}^{\infty} \frac{(a)_{m+p}\,(b)_{m+p}\,x^m z^p}{(a)_m\,(a)_p\,m!\,p!} \sum_{n=0}^{m} \frac{(-m)_n (c)_n}{(a+p)_n\, n!}\left(\frac{y}{y-1}\right)^n. \qquad \ldots(4.2.13)$$

On using the series manipulation identity (3.4.22) in (4.2.13), we get

$$F_P\left[a, a, a; b, c, b; a, a, a; x, y, z\right]$$

$$= (1-y)^{-c} \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}\,(b)_{m+n+p}\,(c)_n\,x^m z^p}{(a)_{m+n}\,(a)_{n+p}\,m!\,n!\,p!}\left(\frac{xy}{1-y}\right)^n$$

$$= (1-y)^{-c} \sum_{n=0}^{\infty} \frac{(b)_n\,(c)_n}{(a)_n\,n!}\left(\frac{xy}{1-y}\right)^n F_4\left[b+n; a+n; a+n, a+n; x, z\right]. \qquad \ldots(4.2.14)$$

Now using a reduction formula of Manocha [68, p.688(line 11)]

$$F_4\left[a; b; b, b; x, y\right] = (1-x-y)^{-a}\;{}_2F_1\left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}\ ; \\ b\ ; \end{matrix}\ \frac{4xy}{(1-x-y)^2}\right], \qquad \ldots(4.2.15)$$

in (4.2.14), we get

$$F_P \left[a, a, a; b, c, b; a, a, a; x, y, z \right]$$

$$= (1-y)^{-c} (1-x-z)^{-b} \sum_{n=0}^{\infty} \frac{(b)_n (c)_n}{(a)_n \; n!} \left(\frac{xy}{(1-y)(1-x-z)} \right)^n$$

$${}_2F_1 \left[\begin{matrix} \frac{b+n}{2}, \frac{b+n+1}{2} ; \\ a+n \quad ; \end{matrix} \frac{4xz}{(1-x-z)^2} \right]$$

$$= (1-y)^{-c} (1-x-z)^{-b} \sum_{n=0}^{\infty} \frac{(b)_n (c)_n}{(a)_n \; n!} \left(\frac{xy}{(1-y)(1-x-z)} \right)^n .$$

$$\sum_{m=0}^{\infty} \frac{(b+n)_{2m}}{(a+n)_m \; m!} \left(\frac{xz}{(1-x-z)^2} \right)^m , \qquad \ldots(4.2.16)$$

which on using the definition (1.5.3) of Horn's function H_3, yields

$$F_F \left[a, a, a; b, c, b; a, a, a; x, y, z \right]$$

$$= (1-y)^{-c} (1-x-z)^{-b} H_3 \left[b, c; a; \frac{xz}{(1-x-z)^2}, \frac{xy}{(1-y)(1-x-z)} \right], \qquad \ldots(4.2.17)$$

and is a correct form of Exton's result (4.1.7). When $z = o$, (4.2.17) reduces to a known reduction formula [7, p.81(7)] .

$$F_2\left[a;\, b,\, c;\, a,\, a;\, x,\, y\right]$$

$$= (1-x)^{-b}\,(1-y)^{-c}\;{}_2F_1\left[\begin{matrix} b,\; c\;; \\ a; \end{matrix}\;\; \frac{xy}{(1-x)(1-y)}\right] . \qquad \ldots(4.2.18)$$

4.3 MORE TRANSFORMATION AND REDUCTION FORMULAE :

In (4.2.1), putting $f = c$, replacing y and z by $y(1-z)$ and $z(1-y)$ respectively, using a transformation of Bailey [8, p.40(4.1); see also 7, p.102 Ex.20(ii)]

$$F_4\left[a;\, b;\, c,\, b;\, \frac{-x}{(1-x)(1-y)},\, \frac{-y}{(1-x)(1-y)}\right]$$

$$= ((1-x)(1-y))^{a}\; F_1\left[a; c-b,\, 1+a-c;\, c;\, x,\, xy\right] \qquad \ldots(4.3.1)$$

and interpreting the result with the help of the definition (1.10.2) of Srivastava's triple series $F^{(3)}$, we get a transformation

$$F_E\left[a,\, a,\, a;\, b,\, c,\, c;\, d,\, e,\, c;\, x,\, y(1-z),\, z(1-y)\right]$$

$$= ((1-y)(1-z))^{-a}\, F^{(3)}\left[\begin{matrix} a :: - ;\, - ;\, 1+a-e : \; b\;; \;\; e-c\,;\, - \,; \\ - :: - ;\, e\,;\, - : d,\, 1+a-e;\; - ;\, - \,; \end{matrix}\;\; \frac{x}{(1-y)(1-z)},\, \frac{-y}{(1-y)},\, \frac{yz}{(1-y)(1-z)}\right] . \qquad \ldots(4.3.2)$$

Now putting $e = c$ in (4.3.2), we get

$$F_E\left[a,a,a;\ b,\ c,\ c;\ d,\ c,\ c;\ x,\ y(1-z),\ z(1-y)\right]$$

$$=((1-y)(1-z))^{-a}\ F^{2:1;0}_{0:2;1}\left[\begin{matrix} a,\ 1+a-c: & b;\ -; \\ \text{———}: & d, 1+a-c;\ c; \end{matrix}\ \frac{x}{(1-y)(1-z)},\ \frac{yz}{(1-y)(1-z)}\right]. \quad \ldots(4.3.3)$$

which further can be reduced in the form of Horn's function G_1 (1.5.1) by setting $d = b$ and using a transformation of Srivastava [107, p.102(1.6)]

$$F_4\left[c+d-1;\ b;\ c,\ d;\ x,y\right] = (1-x-y)^{-b}\ G_1\left[b, 1-c,\ 1-d;\ \frac{x}{1-x-y},\ \frac{y}{1-x-y}\right]. \quad \ldots(4.3.4)$$

Thus we have

$$F_E\left[a,a,a;b,c,c;b,c,c;x,\ y(1-z),\ z(1-y)\right]$$

$$= ((1-y)\ (1-z))^{1-c}\ (1-x-y-z)^{c-a-1}\cdot$$

$$G_1\left[1+a-c,\ c-a,\ 1-c;\ \frac{x}{(1-x-y-z)},\ \frac{yz}{(1-x-y-z)}\right]. \quad \ldots(4.3.5)$$

Again setting $d = b$ in (4.3.3), we get

$$F_E\left[a,a,a;b,c,c;b,c,c;x,y(1-z),z(1-y)\right]$$

$$= ((1-y)(1-z))^{-a} F_4\left[a;1+a-c;\ 1+a-c,\ c;\ \frac{x}{(1-y)(1-Z)},\frac{yz}{(1-y)(1-z)}\right] \qquad \ldots(4.3.6)$$

$$= ((1-y)(1-z))^{-a} \sum_{m=0}^{\infty} \frac{(a)_m}{m!}\left(\frac{x}{(1-y)(1-z)}\right)^m {}_2F_1\left[\begin{matrix} a+m,\ 1+a-c+m; \\ c\ ; \end{matrix}\ \frac{yz}{(1-y)(1-z)}\right]. \qquad \ldots(4.3.7)$$

Now using the result (4.2.18) in (4.3.7) and interpreting the result with the help of the definition (1.1?.1) of another Srivastava's triple series H_B, we get an interesting transformation in the form

$$F_E\left[a,a,a;\ b,\ c,\ c;\ b,\ c,\ c;\ x,\ y(1-z),\ z(1-y)\right]$$

$$= (1-z)^{1-c}\ H_B\left[1+a-c,\ a,c;\ 1+a-c,\ c,\ c;\ x,\ y,\ z\right]. \qquad \ldots(4.3.8)$$

Similarly by suitable adjustment of parameters in (4.2.1) and using the result (4.2.15) of Manocha with the series identity (2.2.1), we get F_E as a combination of two Kampé de Fériet's functions given by

$$F_E\left[a,a,a;\ b,\ c,\ c;\ d,\ c,\ c;\ x,\ y,\ z\right]$$

$$= (1-y-z)^{-a}\ F^{2:2;0}_{0:3;1}\left[\begin{matrix}\frac{a}{2},\frac{a+1}{2}\ :\ \frac{b}{2}\ ,\ \frac{b+1}{2}\ ;\ -\ ;\\ \text{------}:\ \frac{1}{2}\ ,\ \frac{d}{2},\frac{d+1}{2};c\ ;\end{matrix}\right.$$

$$\left.\left(\frac{x}{1-y-z}\right)^2,\ \frac{4yz}{(1-y-z)^2}\right] + \frac{abx}{d(1-y-z)^{a+1}}\ .$$

$$F^{2:2;0}_{0:3;1}\left[\begin{matrix}\frac{a+1}{2},\ \frac{a+2}{2}\ :\ \frac{b+1}{2}\ ,\ \frac{b+2}{2};\ -\ ;\\ \text{------}:\ \frac{3}{2},\ \frac{d+1}{2},\frac{d+2}{2};c\ ;\end{matrix}\ \left(\frac{x}{1-y-z}\right)^2,\ \frac{4yz}{(1-y-z)^2}\right].$$

...(4.3.9)

Similarly with the help of a reduction formula of Bailey [8, p.42(4.3), see also 7, p.102 Ex.20 (iv)]

$$F_4\left[a;\ b;\ b,\ b;\ \frac{-x}{(1-x)(1-y)}\ ,\ \frac{-y}{(1-x)(1-y)}\right]$$

$$= ((1-x)(1-y))^{a}\ {}_2F_1\left[\begin{matrix}a,\ 1+a-b\ ;\\ b\ ;\end{matrix}\ xy\right] \qquad ...(4.3.10)$$

and a result (4.2.18), we get another transformation

$$F_E\left[a,a,a;b,\ c,c;d,c,c;\ x,\ y(1-z),\ z(1-y)\right]$$

$$=(1-z)^{1-c}\ F^{(3)}\left[\begin{matrix}\text{---}\ ::\ a;\ c;\ 1+a-c\ :\ b\ ;\ -;\ -;\\ \text{---}\ ::\ -;\ -;\ \text{----}:\ d,1+a-c;c\ ;\ c;\end{matrix}\ x,y,z\right],$$

...(4.3.11)

which reduces to (4.3.8) for $d = b$.

CHAPTER V

TRANSFORMATION AND REDUCTION FORMULAE OF

SRIVASTAVA'S FUNCTION

5.1 INTRODUCTION :

A number of cases, in which Kampé de Fériet and Appell functions have been shown to reduce to simpler functions for special values of the variables, have appeared in the literature; see for example, Appell and Kampé de Fériet [4], Exton [41], Buschmanand Srivastava [16] and Karlsson [51] et cetera. Recently, Pathan [80], [81], [83], [84], [86] and Khan and Pathan [55], [56] gave a number of transformations and reductions of Srivastava's triple series $F^{(3)}$ by using the Laplace integral representations for the product of Whittaker's functions. The integral of Erdélyi, A. et. al [39,p.216(16)] plays a key role in section 5.2 in obtaining a reduction of Srivastava's triple series $F^{(3)}$ into a combination of four Kampé de Fériet's double series. It is shown how the main result yields a number of well known results of Bailey [6], [10] and Preece [87]. We add a few more entries to the list of reducibility formulas of Kampé de Fériet's function to the known formulas of Burchnall and Srivastava [16, p.439] by obtaining a reduction formula (5.2.18) with the help of Laplace and inverse Laplace techniques.

Just as an integral [35, p.216(16)] was the main tool used in section 5.2 to obtain the transformations, an Wason's integral [75, pp.187, 19] is applied in section 5.3 to obtain the some more transformations and reductions of Srivastava' $F^{(3)}$. In some cases $F^{(3)}$ is transformed into Lauricella's $F_A^{(3)}$ or their combinations. In the sequel we shall, in addition to Appell's functions F_1, F_2, F_3, obtain reductions and transformations to Kampé de Fériet's and other simpler hypergeometric functions.

5.2 USES OF ERDÉLYI'S INTEGRAL :

Using term by term integration with the help of a result of Erdélyi , A. et. al [35, p.216(16)]

$$I_5 = \int_0^\infty e^{-pt}\, t^{(b-1)}\, W_{k,h}(at)\, dt$$

$$= \frac{\Gamma(b+h+\frac{1}{2})\,\Gamma(b-h+\frac{1}{2})\; a^{(h+\frac{1}{2})}}{\Gamma(b-k+1)\;\; (p+\frac{a}{2})^{(b+h+\frac{1}{2})}} .$$

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$$
{}_2F_1\left[\begin{matrix} b+h+\frac{1}{2},\ h-k+\frac{1}{2}\ ; \\ b-k+1 \qquad\qquad ; \end{matrix}\ \frac{p-\frac{a}{2}}{p+\frac{a}{2}}\right], \qquad \ldots(5.2.1)
$$

provided that $Re(b \pm h) > -\frac{1}{2}$ and $Re(p+\frac{a}{2}) > o$, Pathan [81, p. 372(2.1)] has obtained the following integral for a product of three Whittaker's functions

$$
I_6 = \int_o^\infty t^a \exp\left\{\left[-z-\left(\frac{p-x-y}{2}\right)\right]t\right\} W_{k,m}(pt) M_{b,c}(xt)\, M_{g,h}(yt)\,dt
$$

$$
= \frac{x^{c+\frac{1}{2}}\, Y^{h+\frac{1}{2}}\, p^{m+\frac{1}{2}}\ \Gamma(a+c+h+\frac{5}{2}+m)\,\Gamma(a+c+h+\frac{5}{2}-m)}{(z+p)^{(a+c+h+\frac{5}{2}+m)}\ \Gamma(a+c+h+3-k)}
$$

$$
F^{(3)}\left[\begin{matrix} a+c+h+m+\frac{5}{2} :: a+c+h-m+\frac{5}{2} ; -; - : \\ a+c+h-k+3 :: \text{------} ; -; - : \end{matrix}\right.
$$

$$
\left.\begin{matrix} \frac{1}{2}-b+c ; \frac{1}{2}-g+h ; m-k+\frac{1}{2} ; \\ 2c+1 ;\ 2h+1 ; \text{----} ; \end{matrix}\ \frac{x}{z+p}, \frac{y}{z+p}, \frac{z}{z+p}\right], \qquad \ldots(5.2.2)
$$

where $Re(a+c+h+\frac{5}{2} \pm m) > o$, $Re(2z+p-x-y \pm p \pm x \pm y) > o$ and $W_{k,m}(pt)$ and $M_{b,c}(xt)$ are Whittaker's confluent hypergeometric functions given by (3.4.11) and [62, p.274(9.13.16)]

$$M_{b,c}(xt) = (xt)^{\frac{1}{2}+c}\, e^{-\frac{xt}{2}}\ {}_1F_1\left[\begin{matrix}\frac{1}{2}-b+c\ ; \\ 1+2c\ ;\end{matrix}\ xt\right] \qquad \ldots(5.2.3)$$

such that $|\arg xt| < \pi$, respectively.

The special case of Pathan's integral I_6 (5.2.2) can be written in the form

$$I_7 = \int_0^\infty t^B \exp\left\{-(s+\frac{p}{2})t\right\} W_{k,h}(pt)\cdot {}_1F_1\left[\begin{matrix}d\ ; \\ 2e\ ;\end{matrix}\ xt\right]\ {}_1F_1\left[\begin{matrix}d-2e+1\ ; \\ 2-2e\ ;\end{matrix}\ -xt\right]\ dt$$

$$= \frac{\Gamma(B+h+\frac{3}{2})\,\Gamma(B-h+\frac{3}{2})\ p^{(h+\frac{1}{2})}}{\Gamma(B-k+2)\ (s+p)^{B+h+\frac{3}{2}}}\ .$$

$$F^{(3)}\left[\begin{matrix}B+h+\frac{3}{2} :: B-h+\frac{3}{2};\ -;\ -: d;\ d-2e+1\ ;\ h-k+\frac{1}{2}\ ; \\ B-k+2 ::\ \text{———};\ -;\ - : 2e;\ 2-2e\ ;\ \text{———};\end{matrix}\ \frac{x}{s+p},\ \frac{-x}{s+p},\ \frac{s}{s+p}\right], \qquad \ldots(5.2.4)$$

provided that $\operatorname{Re}(B \pm h + \frac{3}{2}) > 0$ and $\operatorname{Re}(s+p) > 0$.

On using the result of Preece [87, p.378(11); see also 64, p.395 (11?(11))]

$$
{}_1F_1\left[\begin{matrix} a; \\ b; \end{matrix}\; x\right] \; {}_1F_1\left[\begin{matrix} a-b+1; \\ 2-b\;; \end{matrix}\; -x\right] = {}_2F_3\left[\begin{matrix} \frac{1+b-2a}{2}, \frac{1-b+2a}{2}; \\ \frac{1}{2}, \frac{1+b}{2}, \frac{3-b}{2}\;; \end{matrix}\; \frac{x^2}{4}\right]
$$

$$
+ \frac{(b-1)(b-2a)x}{b(2-b)} \; {}_2F_3\left[\begin{matrix} \frac{2+b-2a}{2}, \frac{2-b+2a}{2}; \\ \frac{3}{2}, \frac{2+b}{2}, \frac{4-b}{2}\;; \end{matrix}\; \frac{x^2}{4}\right], \quad \ldots.(5.2.5)
$$

in the integral I_7 (5.2.4), expressing each ${}_2F_3$ in power series form, interchanging the order of summations and integration and using the integral I_5 (5.2.1), we have

$$
I_7 = \frac{\Gamma(B+h+\frac{3}{2})\,\Gamma(B-h+\frac{3}{2})\; p^{(h+\frac{1}{2})}}{\Gamma(B-k+2)\;\;(s+p)^{B+h+\frac{3}{2}}} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}+e-d)_n}{(\frac{1}{2})_r}
$$

$$
\frac{(\frac{1}{2}-e+d)_r}{(\frac{1}{2}+e)_r\,(\frac{3}{2}-e)_r\, r!} \left(\frac{x^2}{4(s+p)^2}\right)^r \frac{(B+h+\frac{3}{2})_{2r}\,(B-h+\frac{3}{2})_{2r}}{(B-k+2)_{2r}}
$$

$$
{}_2F_1\left[\begin{matrix} B+h+\frac{3}{2}+2r,\; h-k+\frac{1}{2}\;; \\ B-k+2+2r\;; \end{matrix}\; \frac{s}{s+p}\right] +
$$

$$\frac{(2e-1)(e-d)x\,\Gamma(B+h+\frac{5}{2})\,\Gamma(B-h+\frac{5}{2})\quad p^{(h+\frac{1}{2})}}{2e(1-e)\,\Gamma(B-k+3)\quad (z+p)^{B+h+\frac{5}{2}}}\,.$$

$$\sum_{r=0}^{\infty}\frac{(1+e-d)_r\,(1-e+d)_r}{(\frac{3}{2})_r\,(1+e)_r\,(2-e)_r\,r!}\left(\frac{x^2}{4(z+p)^2}\right)^r.$$

$$\frac{(B+h+\frac{5}{2})_{2r}\quad(B-h+\frac{5}{2})_{2r}}{(B-k+3)_{2r}}\;{}_2F_1\left[\begin{matrix}B+h+\frac{5}{2}+2r,\ h-k+\frac{1}{2}\ ; \\ \\ B-k+3+2r\ ;\end{matrix}\ \frac{z}{z+p}\right].$$

...(5.2.6)

Now using the hypergeometric series identity (2.2.2) in (5.2.6), comparing with (5.2.4) and making suitable adjustment of parameters and variables, we have a reduction of triple series $F^{(3)}$ into a combination of four Kampé de Fériet's double series, in the form

$$F^{(3)}\left[\begin{matrix}2a :: 2b;\ -;\ - : d\ ;\ d-2e+1\ ;\ 2c-2b\ ; \\ \\ 2c :: -;\ -;\ - : 2e;\ 2-2e\ ;\ \text{———};\end{matrix}\ x,\ -x,\ y\right] =$$

$$F^{2:4;2}_{2:3;1}\left[\begin{matrix}a,\ a+\frac{1}{2} : \frac{1}{2}+e-d,\ \frac{1}{2}-e+d, b,\ b+\frac{1}{2};c-b, c-b+\frac{1}{2}; \\ c,\ c+\frac{1}{2} :\ \frac{1}{2},\ \frac{1}{2}+e\quad ,\ \frac{3}{2}-e;\quad \frac{1}{2}\quad ;\end{matrix}\ x^2, y^2\right]$$

$$+\frac{2a(c-b)y}{c}\ .$$

$$F^{2:4;2}_{2:3;1}\left[\begin{matrix} a+\frac{1}{2},\ a+1 : \frac{1}{2}+e-d,\ \frac{1}{2}-e+d,\ b,\ b+\frac{1}{2};c-b+\frac{1}{2},c-b+1; \\ c+\frac{1}{2},\ c+1 : \frac{1}{2},\ \frac{1}{2}+e,\ \frac{3}{2}-e \quad ; \quad \frac{3}{2} \quad ; \end{matrix} x^2, y^2\right]$$

$$+ \frac{ab(2e-1)(e-d)x}{c\, e(1-e)} \cdot$$

$$\left\{ F^{2:4;2}_{2:3;1}\left[\begin{matrix} a+\frac{1}{2},a+1 : 1+e-d,\ 1-e+d,\ b+\frac{1}{2},\ b+1,c-b,c-b+\frac{1}{2}; \\ c+\frac{1}{2},c+1 : \frac{3}{2},\ 1+e,\ 2-e \quad ; \quad \frac{1}{2} \quad ; \end{matrix} x^2, y^2\right]\right.$$

$$+ \frac{2(2a+1)(c-b)y}{(2c+1)} \cdot$$

$$\left. F^{2:4;2}_{2:3;1}\left[\begin{matrix} a+1,a+\frac{3}{2} : 1+e-d,\ 1-e+d,\ b+\frac{1}{2},\ b+1;c-b+\frac{1}{2},c-b+1; \\ c+1,\ c+\frac{3}{2} : \frac{3}{2},\ 1+e,\ 2-e \quad ; \quad \frac{3}{2} \quad ; \end{matrix} x^2, y^2\right]\right\}$$

...(5.2.7)

The following special cases of (5.2.7) are of interest :

When $y = o$ in (5.2.7), we obtain a reduction of

$F^{2:1;1}_{1:1;1}$ into a combination of two ${}_6F_5$'s.

$$F^{2:1;1}_{1:1;1}\left[\begin{matrix} 2a,\ 2b : d ; d-2e+1 ; \\ 2c : 2e ; 2-2e ; \end{matrix} x, -x\right]$$

$$= {}_6F_5\left[\begin{matrix} a,\ a+\frac{1}{2},\ b,\ b+\frac{1}{2},\ \frac{1}{2}+e-d,\ \frac{1}{2}-e+d; \\ \frac{1}{2},\ c,\ c+\frac{1}{2},\ \frac{1}{2}+e,\ \frac{3}{2}-e; \end{matrix}\ x^2\right]$$

$$+ \frac{ab(2e-1)(e-d)x}{ce(1-e)}\ {}_6F_5\left[\begin{matrix} a+\frac{1}{2},\ a+1,\ b+\frac{1}{2},\ b+1,\ 1+e-d,\ 1-e+d; \\ \frac{3}{2},\ c+\frac{1}{2},\ c+1,\ 1+e,\ 2-e; \end{matrix}\ x^2\right].$$

...(5.2.8)

When $e = d$, (5.2.8) reduces to

$$F^{2:1;1}_{1:1;1}\left[\begin{matrix} 2a,\ 2b : d ; 1-d; \\ 2c : 2d; 2-2d; \end{matrix}\ x,\ -x\right]$$

$$= {}_5F_4\left[\begin{matrix} a,\ a+\frac{1}{2},\ b,\ b+\frac{1}{2},\ \frac{1}{2}; \\ c,\ c+\frac{1}{2},\ \frac{1}{2}+d,\ \frac{3}{2}-d; \end{matrix}\ x^2\right]. \qquad ...(5.2.9)$$

Setting $c = b$ in (5.2.7) or (5.2.8), we have a known reduction formula of Appell's function F_2 into a combination of two ${}_4F_3$'s in the form

$$F_2\left[2a; d, d-2e+1; 2e, 2-2e; x, -x\right]$$

$$= {}_4F_3\left[\begin{matrix} a,\ a+\frac{1}{2},\ \frac{1}{2}+e-d,\ \frac{1}{2}-e+d; \\ \frac{1}{2},\ \frac{1}{2}+e,\ \frac{3}{2}-e; \end{matrix}\ x^2\right]$$

$$+ \frac{a(2e-1)(e-d)x}{e(1-e)} \; {}_4F_3\left[\begin{matrix} a+\frac{1}{2}, a+1, & 1+e-d, & 1-e+d; \\ \frac{3}{2} & , 1+e, & 2-e & ; \end{matrix} \; x^2\right], \quad \ldots(5.2.10)$$

which was given by Bailey [10, p.239 (4.6)].

Similarly, we have a reduction formula of $F^{(3)}$ into a combination of two Kampé de Fériet's double series

$$F^{(3)}\left[\begin{matrix} 2a :: 2b; -; -: e; 1-e; 2c-2b; \\ 2c :: --; -; -: 2e; 2-2e; -----; \end{matrix} \; x, -x, y\right]$$

$$= F^{2:3;2}_{2:2;1}\left[\begin{matrix} a, a+\frac{1}{2} : \frac{1}{2}, b, b+\frac{1}{2}; c-b, c-b+\frac{1}{2}; \\ c, c+\frac{1}{2} : \frac{1}{2}+e, \frac{3}{2}-e; \frac{1}{2} ; \end{matrix} \; x^2, y^2\right] + \frac{2a(c-b)y}{c}.$$

$$F^{2:3;2}_{2:2;1}\left[\begin{matrix} a+\frac{1}{2}, a+1 : \frac{1}{2}, b, b+\frac{1}{2}; c-b+\frac{1}{2}, c-b+1; \\ c+\frac{1}{2}, c+1 : \frac{1}{2}+e, \frac{3}{2}-e; \frac{3}{2} ; \end{matrix} \; x^2, y^2\right],$$

$$\ldots(5.2.11)$$

which can be obtained from our main result (5.2.7) when $d = e$.

When $x = o$, (5.2.7) reduces to a particular case of known hypergeometric series identity (2.2.2).

When b = c in (5.2.9) or e = d in (5.2.10), we get another known reduction formula of Bailey [10, p.239(4.4)] in the form

$$F_2\left[a;\ d,\ 1-d;\ 2d,\ 2-2d;\ x,\ -x\right]$$
$$= {}_3F_2\left[\begin{matrix}\frac{1}{2}\ ,\ \frac{a}{2}\ ,\ \frac{a+1}{2}\ ;\\ \\ \frac{1+2d}{2},\ \frac{3-2d}{2}\ ;\end{matrix}\ x^2\right]. \qquad \ldots(5.2.12)$$

On replacing b, a, d and x in (5.2.9) by c, $\frac{a}{2}$, $\frac{d}{2}$ and $\frac{x}{a}$, respectively, taking a → ∞ and using Kummer's first transformation [88, p. 125 (2)]

$${}_1F_1\left[\begin{matrix}a\ ;\\ \\ b\ ;\end{matrix}\ z\right] = e^z\ {}_1F_1\left[\begin{matrix}b-a\ ;\\ \\ b\ ;\end{matrix}\ -z\right]\ ;\ b \neq 0,\ -1,\ -2,\ldots \qquad \ldots(5.2.13)$$

we have a known result of Preece [87, p.375(6.8)]

$${}_1F_1\left[\begin{matrix}\frac{d}{2}\ ;\\ \\ d\ ;\end{matrix}\ x\right]\ {}_1F_1\left[\begin{matrix}\frac{2-d}{2}\ ;\\ \\ 2-d\ ;\end{matrix}\ x\right] = e^x\ {}_1F_2\left[\begin{matrix}\frac{1}{2}\ ;\\ \\ \frac{1+d}{2},\frac{3-d}{2}\ ;\end{matrix}\ \frac{x^2}{4}\right]. \qquad \ldots(5.2.14)$$

Similarly replacing x and e in (5.2.10) by $\frac{x}{2ad}$ and $\frac{e}{2}$, respectively and taking a → ∞, d → ∞, we get a known result of Bailey [6, p.245 (2.05)]

$$ {}_0F_1\left[\begin{matrix} --- ; \\ e ; \end{matrix} x\right] \; {}_0F_1\left[\begin{matrix} --- ; \\ 2-e ; \end{matrix} -x\right] $$

$$ = {}_0F_3\left[\begin{matrix} ------- ; \\ \frac{1}{2}, \frac{2-e}{2}, \frac{e+1}{2} ; \end{matrix} \frac{-x^2}{4}\right] + \frac{2x(1-e)}{e(2-e)} \; {}_0F_3\left[\begin{matrix} ------- ; \\ \frac{3}{2}, \frac{4-e}{2}, \frac{2-e}{2} ; \end{matrix} \frac{-x^2}{4}\right]. \qquad \ldots(5.2.15) $$

For $e = \frac{1}{2}$, (5.2.7) yields a reduction of Srivastava's $F^{(3)}$ into a combination of two Kampé de Fériet's functions

$$ F^{(3)}\left[\begin{matrix} 2a :: 2b; -; -: d; d; 2c-2b; \\ 2c :: -; -; -: 1; 1; ---; \end{matrix} x, -x, y\right] $$

$$ = F^{2:4;2}_{2:3;1}\left[\begin{matrix} a, a+\frac{1}{2} : d, 1-d, b, b+\frac{1}{2}; c-b, c-b+\frac{1}{2} ; \\ c, c+\frac{1}{2} : \frac{1}{2}, 1, 1; \frac{1}{2} ; \end{matrix} x^2, y^2\right] $$

$$ + \frac{2a(c-b)y}{c} \; F^{2:4;2}_{2:3;1}\left[\begin{matrix} a+\frac{1}{2}, a+1 : d, 1-d, b, b+\frac{1}{2}; c-b+\frac{1}{2}, c-b+1 ; \\ c+\frac{1}{2}, c+1 : \frac{1}{2}, 1, 1 ; \frac{3}{2} ; \end{matrix} x^2, y^2\right]. \qquad \ldots(5.2.16) $$

whereas (5.2.8) would give us a reduction of Kampé de Fériet's function into a single hypergeometric series ${}_6F_5$

$$F^{2:1;1}_{1:1;1}\left[\begin{matrix} 2a, 2b : d; & d; \\ & & x, -x \\ 2c : 1; & 1; \end{matrix}\right]$$

$$= {}_6F_5\left[\begin{matrix} a, a+\frac{1}{2}, b, b+\frac{1}{2}, d, 1-d; \\ & x^2 \\ \frac{1}{2}, 1, 1, c, c+\frac{1}{2} ; \end{matrix}\right] . \qquad \ldots(5.2.17)$$

It is to be noted that a more general form of the reduction formula (5.2.8) can be obtained in the following form

$$F^{p:1;1}_{q:1;1}\left[\begin{matrix} (a_p) : d & ; d-2e+1 ; \\ & & x, -x \\ (b_q) : 2e ; & 2-2e ; \end{matrix}\right]$$

$$= {}_{2p+2}F_{2q+3}\left[\begin{matrix} \frac{(a_p)}{2}, \frac{(a_p)+1}{2}, \frac{1}{2}+e-d, \frac{1}{2}+d-e; \\ & 4^{(p-q-1)}x^2 \\ \frac{1}{2}, \frac{(b_q)}{2}, \frac{(b_q)+1}{2}, \frac{1+2e}{2}, \frac{3-2e}{2} ; \end{matrix}\right]$$

$$+ \frac{x(e-d)(2e-1)\prod_{i=1}^{p} a_i}{2e(1-e)\prod_{i=1}^{q} b_i} .$$

$${}_{2p+2}F_{2q+3}\left[\begin{matrix} \frac{(a_p)+1}{2}, & \frac{(a_p)+2}{2}, & 1+e-d, \ 1-e+d\ ; \\ \frac{3}{2}, & \frac{(b_q)+1}{2}, & \frac{(b_q)+2}{2}, 2-e, e+1; \end{matrix}\ 4^{(p-q-1)}x^2\right], \quad \ldots(5.2.18)$$

with the help of Laplace and inverse Laplace transform techniques as stated in the proof of (3.4.36). Here (a_p) denotes the array of p parameters $a_1, a_2, \ldots, a_p$ and (a_0) denotes the absence of the parameter which is shown by a dash in the literature. Using the fact that the empty product is interpreted as unity, for $p = q = c$, (5.2.18), after making suitable adjustment of parameters, gives (5.2.5).

The reduction formula (5.2.18) is an addition to the following known reduction formulae of Buschman and Srivastava [16, p.439(3.4) and (3.5)] for Kampé de Fériet's function $F^{p:1;1}_{q:1;1}$

$$F^{p:1;1}_{q:1;1}\left[\begin{matrix} a_1,\ldots,a_p : b; b ; \\ d_1,\ldots,d_q : e; e ; \end{matrix}\ x, -x\right]$$

$$= {}_{2p+2}F_{2q+3}\left[\begin{matrix} \Delta(2;a_1),\ldots, \Delta(2;a_p), b, e-b; \\ \Delta(2;d_1),\ldots, \Delta(2,d_q), e, \Delta(2;e); \end{matrix}\ 4^{(p-q-1)}x^2\right] \quad \ldots(5.2.19)$$

and

$$F^{p:1;1}_{q:1;1}\left[\begin{matrix} a_1,\ldots, a_p : b; \; e; \\ d_1,\ldots, d_q : 2b; \; 2e; \end{matrix}\; x, -x\right]$$

$$= {}_{2p+2}F_{2q+3}\left[\begin{matrix} \Delta(2;a_1),\ldots, \Delta(2;a_p), \Delta(2;b+e); \\ \Delta(2;d_1),\ldots, \Delta(2;d_q), b+\frac{1}{2}, e+\frac{1}{2}, b+e; \end{matrix}\; 4^{(p-q-1)}x^2\right]. \qquad \ldots(5.2.20)$$

Here $\Delta(n;a)$ abbreviates the array of n parameters

$$\frac{(a+j-1)}{n}, \; j = 1, \ldots, n.$$

For $p = 1$, $q = o$, (5.2.18) reduces to (5.2.19), whereas for $p = 2$, $q = 1$, it reduces to (5.2.8).

When $e = \frac{1}{2}$ in (5.2.18), we have

$$F^{p:1;1}_{q:1;1}\left[\begin{matrix} (a_p) : d; \; d; \\ (b_q) : 1; \; 1; \end{matrix}\; x, -x\right]$$

$$= {}_{2p+2}F_{2q+3}\left[\begin{matrix} \frac{(a_p)}{2}, \frac{(a_p)+1}{2}, d, 1-d; \\ \frac{1}{2}, 1, 1, \frac{(b_q)}{2}, \frac{(b_q)+1}{2}; \end{matrix}\; 4^{(p-q-1)}x^2\right], \qquad \ldots(5.2.21)$$

which is the generalisation of (5.2.17) and is a special case of (5.2.19).

When $e = d$, (5.2.18) reduces to

$$F\begin{matrix} p:1;1 \\ \\ q:1;1 \end{matrix}\left[\begin{matrix} (a_p): d ; 1-d; \\ \\ (b_q): 2d, 2-2d; \end{matrix} \; x, -x\right]$$

$$= {}_{2p+1}F_{2q+2}\left[\begin{matrix} \frac{(a_p)}{2}, \frac{(a_p)+1}{2}, \frac{1}{2} ; \\ \\ \frac{(b_q)}{2}, \frac{(b_q)+1}{2}, \frac{1+2d}{2}, \frac{3-2d}{2}; \end{matrix} \; 4^{(p-q-1)}\, x^2\right], \qquad \ldots(5.2.22)$$

which is the generalization of (5.2.9) and (5.2.12) and is a special case of (5.2.20).

We conclude this section by remarking that, although, some transformation and reduction formulae of Appell's functions F_1 to F_4 and Kampé de Fériet's function are given in Exton's book [41], monograph of Appell and Kampé de Fériet [4], Srivastava [1], [107], [108], Abiodun [1], Bailey [1], Burchnall [13], Carlson [22], [23], Erdélyi [31], [32], Buschman and Srivastava [16], Karlsson [51], Khan and Pathan [56] and Gupta [46], the transformations and reductions obtained above are additional results which have not appeared in the literature.

5.3 USES OF [illegible] INTEGRAL :

Consider the integral in the form

$$I_8 = \int_0^\infty e^{-st}\, t^{(a-1)}\; {}_1F_1\left[\begin{matrix} b\,; \\ c\,; \end{matrix}\; xt\right]\; {}_1F_1\left[\begin{matrix} h\,; \\ k\,; \end{matrix}\; zt\right]$$

$$.\,\Psi'(f,\, g;\, yt)\, dt\,, \qquad \ldots(5.3.1)$$

where $\Psi(f,\, g;\, yt)$ is another type of confluent hypergeometric function introduced in 1927 by F. G. Tricomi [33, pp.257(3,7), 262(1); see also 62, p.270 (9.12.2)] in the form

$$\lim_{c \to \infty}\; {}_2F_1\left[\begin{matrix} f,\ f-g+1\,; \\ c\,; \end{matrix}\; 1-(\frac{c}{yt})\right]$$

$$= {}_2F_0\left[\begin{matrix} f,\ f-g+1\,; \\ \text{———}\,; \end{matrix}\; \frac{-1}{yt}\right]$$

$$= (yt)^f\; \Psi(\, f,\, g;\, yt\,)$$

$$= \frac{\Gamma(1-g)\,(yt)^f}{\Gamma(\,f-g+1\,)}\; {}_1F_1\left[\begin{matrix} f; \\ g; \end{matrix}\; yt\right] + \frac{\Gamma(g-1)(yt)^{(1-g+f)}}{\Gamma f}\; {}_1F_1\left[\begin{matrix} f-g+1; \\ 2-g; \end{matrix}\; yt\right]$$

$$\ldots(5.3.2)$$

$$= (yt)^{f+1}\,\Psi'(f+1,g+1;yt)+(yt)^{f}\,(1+f-g)\,\Psi(f+1,g;yt), \qquad \ldots(5.3.3)$$

such that $|\arg(yt)| < \pi,\ g \neq 0, \pm 1, \pm 2, \ldots$.

Now in the right hand side of (5.3.1), expressing ${}_1F_1\left[\begin{matrix} h; \\ k; \end{matrix}\ gt\right]$ in power series form, interchanging the order of summation and integration, integrating term by term with the help of Gleson's integral [76, pp.1286(15), 1287(16); see also 2, p.2018(18) and 75, pp.187(1a), 190(8)].

$$I_9 = \int_0^\infty e^{-st}\, t^{(a-1)}\, {}_1F_1\left[\begin{matrix} b\ ; \\ c\ ; \end{matrix}\ xt\right] \Psi'(f,\ g;\ yt)\ dt$$

$$= \frac{\Gamma a\ \Gamma(a-g+1)}{\Gamma(a-g+1+f)\ y^a} \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\ (a-g+1)_{m+n}\ (b)_m}{(a-g+1+f)_{m+n}\ (c)_m\ m!\ n!}$$

$$\left(\frac{x}{y}\right)^m \left(\frac{y-s}{y}\right)^n , \qquad \ldots(5.3.4)$$

provided that $Re(a-g+1) > 0$, $|\,I_m(x)| < 1$, $Re(s) > 0$;

using Euler's first linear transformation (4.2.12) and interpreting the result with the help of the definition of Srivastava's triple series $F^{(3)}$, we get

$$I_8 = \frac{\Gamma a\ \Gamma(a-g+1)}{\Gamma(a-g+1+f)\ \ s^a} \ .$$

$$F^{(3)}\left[\begin{matrix} a :: -; -; a-g+1 : b; f; h; \\ a-g+1+f :: -; -; ---- : c; ----; k; \end{matrix} \frac{x}{s}, \frac{s-y}{s}, \frac{z}{s}\right]. \quad ...(5.3.5)$$

On using Kummer's first transformation (5.2.13) for both ${}_1F_1$'s in the right hand side of the integral I_8, expressing one ${}_1F_1$ in power series form and again using Olsson's integral I_9, we get

$$I_8 = \frac{\Gamma a\ \Gamma(a-g+1)}{\Gamma(a-g+1+f)y^a}.$$

$$F^{(3)}\left[\begin{matrix} a, a-g+1 :: -; -; -: c-b; -; k-h; \\ a-g+1+f :: -; -; -: c; -; k; \end{matrix} \frac{-x}{y}, \frac{x+y+z-s}{y}, \frac{-z}{y}\right].$$

$$...(5.3.6)$$

On using (5.3.2) for $\psi(f,g;yt)$ in I_8 (5.3.1), expressing each ${}_1F_1$'s in power series form, integrating with the help of the definition (2.2.18) of Gamma function and interpreting the result with the help of the definition (1.7.1) of Lauricella's triple hypergeometric function $F_A^{(3)}$, we get

$$I_8 = \frac{\Gamma(1-g)\ \Gamma a}{\Gamma(f-g+1)\ (s-y)^a}\ F_A^{(3)}\left[a; b, g-f, h; c, g, k; \frac{x}{s-y}, \frac{y}{y-s}, \frac{z}{s-y}\right]$$

$$+ \frac{\Gamma(g-1)\ y^{(1-g)}\Gamma(a+1-g)}{\Gamma f\ \ (s-z)^{(a+1-g)}}.$$

$$F_A^{(3)}\left[a+1-g;b,\ f-g+1,k-h;\ c,\ 2-g,\ k;\ \frac{x}{s-z},\ \frac{y}{s-z},\frac{z}{z-s}\right]. \quad \ldots(5.3.7)$$

On the other hand, if we use the result (5.3.3) in the integral I_8 (5.3.1), apply again Olsson's integral I_9(5.3.4) with the Euler's first (4.2.12) and second linear transformation [62, p.248(9.5.3)] given by

$${}_2F_1\left[\begin{matrix}a,\ b;\\ \\ c\ ;\end{matrix}\ x\right]=(1-x)^{c-a-b}\ {}_2F_1\left[\begin{matrix}c-a,c-b;\\ \\ c\ ;\end{matrix}\ x\right];\ |\arg(1-x)|<\pi,$$

and interpret the result in the form of $F^{(3)}$, we get

$$I_8=\frac{\Gamma(a+1)\,\Gamma(a+1-g)\,y^{(1-g)}}{\Gamma(a+2+f-g)}\,F^{(3)}\left[\begin{matrix}a+1-g::-;\ -;\ a+1: & b;\ f-g+1;\ h; \\ a+2+f-g::-;-;-: & c;\ -\!-\!-\ ;\ k;\end{matrix}\ \frac{x}{s},\frac{s-y}{s},\ \frac{z}{s}\right]+\frac{(1+f-g)\Gamma a\ \Gamma(a-g+1)}{y^{(1+f)}}\,\frac{s^{(1+f-a)}}{\Gamma(a+2-g+f)}$$

$$.F^{(3)}\left[\begin{matrix}-\!-\!-\!-\!-::-;\ -;\ a-g+1,a:b\ ;\ 2+f-g,1+f;\ h;\\ a-g+2+f\ ::-;\ -;\ -\!-\!-\!-:c\ ;\ -\!-\!-\!-\!-;\ k;\end{matrix}\ \frac{x}{s},\frac{y-s}{y},\frac{z}{s}\right]. \quad \ldots(5.3.8)$$

Now comparing the integrals (5.3.5), (5.3.6), (5.3.7) and (5.3.8) and making suitable adjustment of parameters and variables, we get the following three transformations

$$F^{(3)}\left[\begin{array}{l} a :: -;\ -;\ a-g+1 : b;\ f;\ h; \\ \qquad\qquad\qquad\qquad\qquad x,\ 1-y,\ z \\ a-g+1+f :: -;\ -;\ \text{----} : c;\ -;\ k; \end{array}\right]$$

$$= y^{-a}\, F^{(3)}\left[\begin{array}{l} a, a-g+1 :: -;\ -;\ - : c-b;\ -;\ k-h; \\ a-g+1+f :: -;\ -;\ - : c;\ -;\ k; \end{array} \frac{-x}{y}, \frac{x+y+z-1}{y}, \frac{-z}{y}\right],$$

...(5.3.9)

$$= \frac{(f-g+1)_a}{(1-g)_a (1-y)^a}\, F_A^{(3)}\left[a; b, g-f, h; c, g, k; \frac{x}{1-y}, \frac{y}{y-1}, \frac{z}{1-y}\right] +$$

$$\frac{(a)_{g-a-1}\,(f)_{a-g+1}}{(1-z)^{(a-g+1)}\, y^{(g-1)}}\; F_A^{(3)}\left[a+1-g; b, f-g+1, k-h; c, 2-g, k; \frac{x}{1-z}, \frac{y}{1-z}, \frac{z}{z-1}\right],$$

...(5.3.10)

$$= \frac{a\ y^{(1-g)}}{(a-g+1+f)}\, F^{(3)}\left[\begin{array}{l} a+1-g :: -;-;a+1 : b; f-g+1;\ h; \\ \qquad\qquad\qquad\qquad\qquad x, 1-y, z \\ a+2+f-g :: -;-;\text{---} : c; \text{----};\ k\ ; \end{array}\right] +$$

$$\frac{(1+f-g)}{y^{(1+f)}(1+a-g+f)}\, F^{(3)}\left[\begin{array}{l} \text{----} :: -;\ -;\ a-g+1,\ a : \\ a-g+2+f :: -;\ -;\ \text{----} : \end{array}\right.$$

$$\left.\begin{array}{l} b;\ 2-g+f, 1+f;\ h; \\ \qquad\qquad\qquad x,\ \frac{y-1}{y},\ z \\ c;\ \text{----};\ k, \end{array}\right],$$

...(5.3.11)

where the parameters and variables are restricted in such a manner that the two sides are meaningful in every case.

Now we shall discuss some particular cases of transformation formulas given above.

When $x = o$ and $1 + a - g = d$, (5.3.9) reduces to a transformation of Kampé de Fériet's function, in the form

$$F\begin{matrix} 2:o;1 \\ \\ 1:o;1 \end{matrix}\left[\begin{matrix} a, d : - ; k-h ; \\ \\ d+f : - ; k ; \end{matrix} \;\; \frac{y+z-1}{y}, \frac{-z}{y}\right]$$

$$= y^{a} \; F\begin{matrix} 1:1;2 \\ \\ 1:o;1 \end{matrix}\left[\begin{matrix} a : f; d, h; \\ \\ d+f : - ; k ; \end{matrix} \;\; 1-y, z\right] . \qquad \ldots(5.3.12)$$

For $y = 1$, (5.3.12) reduces to a reduction formula

$$F\begin{matrix} 2:o;1 \\ \\ 1:o;1 \end{matrix}\left[\begin{matrix} a, d : - ; k-h ; \\ \\ d+f : - ; k ; \end{matrix} \;\; z, -z\right]$$

$$= {}_3F_2\left[\begin{matrix} a, d, h ; \\ \\ d+f, k ; \end{matrix} \;\; z\right] . \qquad \ldots(5.3.13)$$

On setting $f = o$ and applying Binomial theorem

[88, p.74(3)], (5.3.13) reduces to known transformation (4.2.12).

For $x = z = o$, (5.3.9) reduces to (4.2.12).

When $y = 1$ and $f = o$, (5.3.9) reduces to

$$F^{(3)}\left[\begin{matrix} a ::-;-;-: c-b; -; k-h; \\ - ::-;-;-: c ; -; k ; \end{matrix} -x, x+z, -z\right]$$

$$= F_2 \left[a; b, h; c, k; x, z\right]. \qquad \ldots(5.3.14)$$

Setting $x = z = o$, replacing y by $1-y$, g by $1+a+f-g$ in (5.3.10) and using (4.2.12), we get a known result [62, p.249 (9.5.7)]

$${}_2F_1\left[\begin{matrix} a, f ; \\ g ; \end{matrix} y\right] = \frac{\Gamma g\, \Gamma(g-a-f)}{\Gamma(g-a)\Gamma(g-f)}\, {}_2F_1\left[\begin{matrix} a, f ; \\ 1+a+f-g; \end{matrix} 1-y\right] +$$

$$\frac{\Gamma(a+f-g)\,\Gamma g}{\Gamma a\, \Gamma f} (1-y)^{g-a-f}\, {}_2F_1\left[\begin{matrix} g-f, g-a; \\ 1-a-f+g ; \end{matrix} 1-y\right], \qquad \ldots(5.3.15)$$

such that $|\arg y| < \pi$, $|\arg(1-y)| < \pi$ and $a+f-g \neq 0, \pm 1, \pm 2, \pm 3, \ldots$

When c = b and g = 1+ a− d in (5.3.9), we have

$$F^{(3)}\left[\begin{array}{l} a\ ::\ -;-;\ d\ :\ -;\ f;\ h\ ; \\ \qquad\qquad\qquad\qquad\qquad x,\ y,\ z \\ d+f::\ -;-;\ -\ :\ -;\ -;\ k\ ; \end{array}\right]$$

$$=(1-y)^{-a}\ F^{2:0;1}_{1:0;1}\left[\begin{array}{l} a,\ d\ :\ -;\ k-h\ ; \\ d+f\ :-;\ \ k\ \ ; \end{array}\ \frac{x-y+z}{1-y},\ \frac{z}{y-1}\right]. \qquad ...(5.3.16)$$

For k = h, (5.3.16) reduces to

$$F^{(3)}\left[\begin{array}{l} a\ ::\ -;\ -;\ d\ :\ -;\ f\ ;\ -\ ; \\ \qquad\qquad\qquad\qquad\qquad x,\ y,\ z \\ d+f\ ::\ -;\ -;\ -\ :\ -;\ -;\ -\ ; \end{array}\right]$$

$$=(1-y)^{-a}\ {}_2F_1\left[\begin{array}{l} a\ ,\ d\ ; \\ \qquad\qquad \frac{x-y+z}{1-y} \\ d+f\ \ ; \end{array}\right], \qquad ...(5.3.17)$$

which, further for z = o, reduces to a known reduction formula [7, p.79(2)]

$$F_1\ \left[a;d,f;d+f;x,y\right] = (1-y)^{-a}\ {}_2F_1\left[\begin{array}{l} a,\ d; \\ \qquad\qquad \frac{x-y}{1-y} \\ d+f; \end{array}\right]. \qquad ...(5.3.18)$$

When k = h and y = o , (5.3.16) reduces to a known result

[14, p.25?; see also 97, p.21?(8.1.2)]

$$F^{2:0;0}_{1:0;0}\left[\begin{matrix} a,d : -; -; \\ d+f : -; -; \end{matrix} x, z\right] = {}_2F_1\left[\begin{matrix} a, d ; \\ d+f ; \end{matrix} x+z\right]. \quad \ldots(5.3.19)$$

When b = o or x = o and g = 1+ a-d in (5.3.10), we have

$$F^{1:1;2}_{1:0;1}\left[\begin{matrix} a : f; h, d; \\ d+f: -; k ; \end{matrix} y, z\right]$$

$$= \frac{(f+d-a)_a}{y^a(d-a)_a} F_2\left[a; 1+a-d-f, h; 1+a-d, k; \frac{y-1}{y}, \frac{z}{y}\right]$$

$$+ \frac{(a)_{d-2}(f)_d}{(1-y)^{a-d}(1-z)^d} F_2\left[d; f+d-a, k-h; 1-a+d, k; \frac{y-1}{y}, \frac{z}{y}\right].$$

$$\ldots(5.3.20)$$

Setting z = o in (5.3.20) and making suitable adjustment of parameters and variables, we get again (5.3.15).

When x = o or b = o in (5.3.11), we get

$$F^{1:1;2}_{1:0;1}\left[\begin{matrix} a : f ; h, d; \\ d+f: - ; k ; \end{matrix} 1-y, z\right]$$

$$= \frac{a\, y^{(d-a)}}{(d+f)} F^{1:1;2}_{1:0;1}\left[\begin{matrix} d : f-a+d; h, a+1; \\ d+1+f : ——; k ; \end{matrix} 1-y, z\right]$$

$$+ \frac{(f-a+d)}{y^{(1+f)}(d+f)} \; F^{\,o:2;3}_{\,1:o;1} \left[\begin{matrix} \text{-----}: 1+f,\ 1+f-a+d; d, a, h; \\ d+1+f: \text{-----------}; \quad k \quad ; \end{matrix} \ \frac{y-1}{y},\ z \right].$$

...(5.3.21)

When k = h in (5.3.21), we have

$$F_3 \left[1+f-a+d,\ a;\ 1+f,\ d;\ d+1+f;\ \frac{y-1}{y},\ z \right]$$

$$= \frac{y^{(1+f)}(d+f)}{(f-a+d)} \; F_1 \left[a;\ f,\ d;\ d+f;\ 1-y,\ z \right]$$

$$- \frac{a\, y^{(1+d+f-a)}}{(f-a+d)} \; F_1 \left[d;\ f-a+d,\ a+1;\ d+1+f;\ 1-y,\ z \right].$$

...(5.3.22)

CHAPTER VI

TRANSFORMATION AND REDUCTION FORMULAE OF EXTON'S FUNCTIONS

6.1 INTRODUCTION :

H. Exton, whose work [41] on hypergeometric functions of one and more variables has become a legend in presenting systematically the most recent and uptodate developments on the theoretic and applicative aspects of these functions. The present chapter is devoted to his important functions of three variables $(k)_{H_4}(n)$ and $(k)_{H_3}(n)$ (for $k = 1$, $n = 3$) and functions K_1, K_2 and K_3 of four variables. We establish here certain transformation and reduction formulae of these functions into functions of Srivastava, Kampé de Fériet, Appell, Horn, Lauricella, Carlson, Pathan and Exton.

In addition to the transformation and reduction formulae for hypergeometric series of two and three variables obtained in the work of Pathan [78], [79], [85], Bhatt [12], Pandey [77], Extón [41], [42, p.339(13)], Srivastava [107]; Khan and Pathan [55, see also 54] gave some formulae of this kind for Exton's series ${}^{(1)}H_4^{(3)}$. In [55], an integral representation of ${}^{(1)}H_4^{(3)}$ was used to obtain the main transformation formula. Motivated by this formula, in section 6.2, we would obtain a number of transformations and reductions of ${}^{(1)}H_4^{(3)}$ function by using known transformations and series manipulations. Since ${}^{(1)}H_4^{(3)}$ is a generalisation of Horn's function H_4 and Appell's function of second kind F_2, many formulae of H_4 and F_2 follow as special cases of our main results. Many more connections of ${}^{(1)}H_4^{(3)}$ to functions of Srivastava's $F^{(3)}$, Kampé de Fériet, Exton's K_{10} and their special cases are also discussed.

Section 6.3 is a sequel to the previous section and it aims at deriving some transformations and reductions of Exton's function ${}^{(1)}H_3^{(3)}$ which include as their special cases such hypergeometric functions as Lauricella's $F_D^{(3)}$, Saran's F_S, Pathan's $F_P^{(4)}$ and Carlson's R. It is also shown how the main results (6.3.3) and (6.3.8) are related to a number of known results for Appell's F_3 and Horn's H_3 functions given in the work of Erdélyi [31], [32].

Sections 6.4 to 6.6 deal with the reducibility of Exton's functions of four variables K_1, K_2 and K_3 to Lauricella's $F_G^{(3)}$, Saran's F_G and Srivastava's $F^{(3)}$ functions. Some results of Saran's F_E, F_P and Appell's F_2, F_4 are deduced as special cases.

This chapter concludes by presenting in section 6.7, a correction to a transformation of Exton's function ${}^{(k)}H_4^{(n)}$ given in [41, p.127]. For this purpose we shall be making use of the Laplace integral representation [41, p.104] for ${}^{(k)}H_4^{(n)}$.

6.2 TRANSFORMATIONS OF ${}^{(1)}H_4^{(3)}$ - FUNCTION :

Beginning with a particular case of Exton's generalised Horn function ${}^{(k)}H_4^{(n)}$ (1.12.7) for k = 1, n = 3, we may write

$$
{}^{(1)}H_4^{(3)}\left[a, b, c; d, e, f; x, y, z\right]
$$

$$
= \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p}\,(b)_n\,(c)_p\, x^m\, y^n\, z^p}{(d)_m\,(e)_n\,(f)_p\, m!\, n!\, p!}
$$

$$
= \sum_{n,p=0}^{\infty} \frac{(a)_{n+p}(b)_n(c)_p\, y^n z^p}{(e)_n\,(f)_p\, n!\, p!}\; {}_2F_1\left[\begin{matrix} \frac{a+n+p}{2}, \frac{a+n+p}{2}+\frac{1}{2}; \\ d\;; \end{matrix}\; 4x\right]. \qquad \ldots(6.2.1)
$$

Now using a Goursat's quadratic transformation [33, p.112(17)]

$$ {}_2F_1\left[\begin{matrix} a,\ a+\frac{1}{2}\ ; \\ \\ c\ ; \end{matrix}\ z\right] = (1+z^{1/2})^{-2a}\ {}_2F_1\left[\begin{matrix} 2a, c-\frac{1}{2}\ ; \\ \\ 2c-1\ ; \end{matrix}\ \frac{2z^{1/2}}{(1+z^{1/2})}\right], \qquad \ldots(6.2.2) $$

in (6.2.1) and interpreting the result in the form of Srivastava's $F^{(3)}$, we get

$$ {}^{(1)}H_4^{(3)}\left[a, b, c; d, e, f; x^2, y, z\right] $$

$$ = \left(\frac{1}{1+2x}\right)^{a} F^{(3)}\left[\begin{matrix} a::-;\ -,-:\ d-\frac{1}{2};\ b;\ c; \\ \\ -::-;\ -;-:\ 2d-1;\ e;f\ ; \end{matrix}\ \frac{4x}{1+2x}, \frac{y}{1+2x}, \frac{z}{1+2x}\right]. \qquad \ldots(6.2.3) $$

If we replace x by $(\frac{x}{2}) - (\frac{x}{2})^2$ in (6.2.1) and use a result [33, p.111(6)]

$$ {}_2F_1\left[\begin{matrix} a,\ a+\frac{1}{2}\ ; \\ \\ b\ ; \end{matrix}\ 2z-z^2\right] = (1-\frac{z}{2})^{-2a}\ {}_2F_1\left[\begin{matrix} 2a,\ 2a-b+1\ ; \\ \\ b\ ; \end{matrix}\ (\frac{z}{2-z})\right], \qquad \ldots(6.2.4) $$

we get

$$ {}^{(1)}H_4^{(3)}\left[a, b, c; d, e, f; (\tfrac{x}{2}) - (\tfrac{x}{2})^2, y, z\right] $$

$$=\left(\frac{2}{2-x}\right)^{d} F^{(3)}\left[\begin{matrix} a-d+1,\ a::-;\ \text{———};\ -:\ -;\ b;\ c; \\ \text{———}\ ::-;\ a-d+1;\ -:\ d;\ e;\ f; \end{matrix}\ \left(\frac{x}{2-x}\right),\ \left(\frac{2y}{2-x}\right),\ \left(\frac{2z}{2-x}\right)\right]. \qquad \ldots(6.2.5)$$

Next, we consider

$$^{(1)}H_4^{(3)}\left[a,\ b,\ c;\ d,\ e,\ 2c;\ x,\ y,\ z\right]$$

$$=\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n\,y^n}{(e)_n\ \ n!}\ H_4\left[a+n,\ c;\ d,\ 2c;\ x,\ z\right] \qquad \ldots(6.2.6)$$

and use a transformation of Erdélyi [31, p.382(7.6) ; see also 32]

$$H_4\left[a,\ b;\ c,\ 2b;\ x,\ y\right]$$

$$=\left(1-\frac{y}{2}\right)^{-a}\ F_4\left[\frac{a}{2},\ \frac{a+1}{2};\ c,\ b+\frac{1}{2};\ \frac{16x}{(2-y)^2},\ \frac{y^2}{(2-y)^2}\right] \qquad \ldots(6.2.7)$$

to get

$$^{(1)}H_4^{(3)}\left[a,\ b,\ c;\ d,\ e,\ 2c;\ x,\ y,\ z\right]$$

$$=\left(\frac{2}{2-z}\right)^{a}\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(e)_n\ n!}\left(\frac{2y}{2-z}\right)^{n} F_4\left[\frac{a+n}{2},\ \frac{a+n+1}{2};\ d,\ \frac{2c+1}{2};\ \frac{16x}{(2-z)^2},\ \left(\frac{z}{2-z}\right)^2\right]. \qquad \ldots(6.2.8)$$

Now using the series identity (2.2.1) in (6.2.8) and interpreting the result in the form of $F^{(3)}$, we get a reduction of ${}^{(1)}H_4^{(3)}$ into a combination of two $F^{(3)}$'s in the form

$$
\begin{aligned}
&{}^{(1)}H_4^{(3)}\left[a,\ b,\ c;\ d,\ e,\ 2c,\ x,\ y,\ z\right]\\
&= \left(\frac{2}{2-z}\right)^{a} F^{(3)}\left[\begin{matrix}\frac{a}{2},\frac{a+1}{2}::-;\ -;\ -:\ -;\ \frac{b}{2},\ \frac{b+1}{2};\ ----;\\ ----::-;-;\ -:\ d;\ \frac{1}{2},\ \frac{e}{2},\frac{e+1}{2};\frac{2c+1}{2};\end{matrix}\right.\\
&\left.\frac{16x}{(2-z)^2},\left(\frac{2y}{2-z}\right)^2,\left(\frac{z}{2-z}\right)^2\right] + \frac{aby}{e}\left(\frac{2}{2-z}\right)^{a+1} F^{(3)}\left[\begin{matrix}\frac{a+1}{2},\ \frac{a+2}{2}::\\ -------::\end{matrix}\right.\\
&\left.\begin{matrix}--;\ --;\ --:\ --;\ \frac{b+1}{2},\ \frac{b+2}{2};\ ----;\\ --;\ --;\ --:\ d;\ \frac{3}{2},\ \frac{e+1}{2},\frac{e+2}{2};\ \frac{2c+1}{2};\end{matrix}\ \frac{16x}{(2-z)^2},\left(\frac{2y}{2-z}\right)^2,\left(\frac{z}{2-z}\right)^2\right].
\end{aligned}
$$

...(6.2.9)

Now consider the series in the form

$$
\begin{aligned}
&{}^{(1)}H_4^{(3)}\left[a,\ b,\ b;\ d,\ 2b,\ 2b;\ x,\ 2y,\ 2z\right]\\
&= \sum_{m=0}^{\infty}\frac{(a)_{2m}\,x^m}{(d)_m\ m!}\ F_2\left[a+2m;\ b,\ b;\ 2b,\ 2b;\ 2y,\ 2z\right]
\end{aligned}
$$

...(6.2.10)

and using the result (4.2.5) of Bailey and a result of

Carlson [22, p.961; see also 96, p.34 (3.10)]

$$F_4\left[a; b; c, c; x^2, y^2\right]$$

$$= F^{2:1;1}_{2:0;0}\left[\begin{matrix} a, b & : c-\frac{1}{2}; c-\frac{1}{2}; \\ c, 2c-1 : & ---; ---; \end{matrix} (x+y)^2, (x-y)^2\right], \quad \ldots(6.2.11)$$

we get a transformation of ${}^{(1)}H_4^{(3)}$ into $F^{(3)}$

$${}^{(1)}H_4^{(3)}\left[a, b, b; d, 2b, 2b; x, 2y, 2z\right]$$

$$=\left(\frac{1}{1-y-z}\right)^a F^{(3)}\left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2} :: --; ------; --: - ; b; b, \\ ------:: --; 2b, \frac{2b+1}{2}; --: d ; -, - ; \end{matrix}\right.$$

$$\left.\frac{4x}{(1-y-z)^2}, \left(\frac{y+z}{1-y-z}\right)^2, \left(\frac{y-z}{1-y-z}\right)^2\right]. \quad \ldots(6.2.12)$$

Similarly by making suitable adjustment of parameters and variables in (6.2.1) and using the transformation [88, p.66(2)]

$${}_2F_1\left[\begin{matrix} a, b ; \\ 1+a-b; \end{matrix} z\right] = \left(\frac{1}{1+z}\right)^a {}_2F_1\left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2} ; \\ 1+a-b; \end{matrix} \frac{4z}{(1+z)^2}\right] \quad \ldots(6.2.13)$$

and the result (4.2.15) of Manocha, we have the following transformations

$$^{(1)}H_4^{(3)}\left[a,\ b,\ c;\ d,\ e,\ f;\ \frac{x}{(1+x)^2},\ y,\ z\right]$$

$$=(1+x)^{a}\ F^{(3)}\left[\begin{matrix} a,a-d+1 :: -;\ -;\ -:\ -;\ b;\ c\ ; \\ \text{———} :: -;\ a-d+1;-:\ d\ ;\ e,\ f\ ; \end{matrix}\ x,y(1+x),z(1+x)\right],$$

...(6.2.14)

$$E_{10}\left[a,a,a,a;d,d,b,c;d,d,e,f;x,t,y(1-x-t),z(1-x-t)\right]=(1-x-t)^{-a}.$$

$$^{(1)}H_4^{(3)}\left[a,b,\ c;\ d,e,f;\ \frac{xt}{(1-x-t)^2},\ y,\ z\right],\qquad ...(6.2.15)$$

where E_{10} is Exton's quadruple hypergeometric function given by the equation (1.12.4).

Similarly we get a reduction formula of $^{(1)}H_4^{(3)}$ into Kampe de Feriet's function in the form

$$^{(1)}H_4^{(3)}\left[a,\ b,\ e-b;\ d,\ e,\ e;\ x,\ y,\ y\right]$$

$$=(1-y)^{-a}\ F^{2:0;2}_{0:1;3}\left[\begin{matrix} \frac{a}{2},\frac{a+1}{2} :\ -;\ b,\ e-b\ ; \\ \text{———} :d\ ;e,\frac{e}{2},\frac{e+1}{2}\ ; \end{matrix}\ \frac{4x}{(1-y)^2},\left(\frac{y}{1-y}\right)^2\right].$$

...(6.2.16)

Now we shall deduce some particular cases of above transformation and reduction formulae.

When $z = o$ or $c = o$, (6.2.3) reduces to a known transformation of Erdélyi [31, p.381 (7.4); see also 111, p.42(29)]

$$(1+2x)^{a}\ H_4\left[a,\ b;\ d,\ e;\ x^2,\ y\right]$$

$$= F_2\left[a;\ d-\tfrac{1}{2},\ b;\ 2d-1,\ e;\ \frac{4x}{1+2x},\ \frac{y}{1+2x}\right], \qquad \ldots(6.2.17)$$

where H_4 is Horn's functions given by the equation (1.5.4).

When $b = o$ or $y = o$ and $f = o$ in (6.2.5), we get

$$\left(\frac{2-x}{2}\right)^{a}\ H_4\left[a,\ o;\ d,\ c;\ \left(\tfrac{x}{2}\right)-\left(\tfrac{x}{2}\right)^2,\ z\right]$$

$$= F_4\left[a;\ a-d+1,\ d,\ a-d+1;\ \frac{x}{2-x},\ \frac{2z}{2-x}\right]. \qquad \ldots(6.2.18)$$

When $z = o$, $e = b$, (6.2.9) reduces to

$$H_4\left[a,\ b;\ d,\ b;\ x,\ y\right] = F_4\left[\frac{a}{2};\frac{a+1}{2};\ d,\ \frac{1}{2};\ 4x,\ y^2\right]$$

$$+\ ay\ F_4\left[\frac{a+1}{2},\ \frac{a+2}{2};\ d,\ \frac{3}{2};\ 4x,\ y^2\right]. \qquad \ldots(6.2.19)$$

When $x = o$, $e = b$ in (6.2.9), we get

$$F_2\left[a;\ b,\ c;\ b,\ 2c;\ y,\ z\right] = \left(\frac{2}{2-z}\right)^a F_4\left[\frac{a}{2};\frac{a+1}{2};\frac{1}{2},\ \frac{2c+1}{2};\right.$$

$$\left.\left(\frac{2y}{2-z}\right)^2,\left(\frac{z}{2-z}\right)^2\right] + ay\left(\frac{2}{2-z}\right)^{a+1} F_4\left[\frac{a}{2},\ \frac{a+1}{2};\ \frac{3}{2},\ \frac{2c+1}{2};\left(\frac{2y}{2-z}\right)^2,\right.$$

$$\left.\left(\frac{z}{2-z}\right)^2\right]. \qquad \ldots(6.2.20)$$

When $z = y$, (6.2.12) reduces to

$$^{(1)}H_4^{(3)}\left[a,\ b,\ b;\ d,\ 2b,\ 2b;\ x,\ 2y,\ 2y\right]$$

$$=\left(\frac{1}{1-2y}\right)^a .\ F\begin{matrix}2;0;1\\ \\ 0;1;2\end{matrix}\left[\begin{matrix}\frac{a}{2},\ \frac{a+1}{2}\ :\ --;\quad b\quad ;\\ \\ \text{------}\ :\ d\ ;\ 2b,\ b+\frac{1}{2}\ ;\end{matrix}\ \frac{4x}{(1-2y)^2},\right.$$

$$\left.\left(\frac{2y}{1-2y}\right)^2\right], \qquad \ldots(6.2.21)$$

which is also a particular case of the reduction formula (6.2.16).

When $z = o$ or $c = o$ and $e = b$ in (6.2.14), we have

$$H_4\left[a,b;d,b;\ \frac{x}{(1+x)^2},y\right] = (1+x)^a\ F_4\left[a;a-d+1;d,a-d+1;x,y(1+x)\right]. \qquad \ldots(6.2.22)$$

Setting $z = o$ and replacing y by $\left(\frac{y}{1-x-t}\right)$ in (6.2.15), we get a known reduction formula of Exton [41,p.116(4.1.16)]

$$F_E\left[a, a, a; b, d, d; e, d, d; y, x, t\right]$$

$$=(1-x-t)^{-a} H_4\left[a, b; d, e; \frac{xt}{(1-x-t)^2}, \left(\frac{y}{1-x-t}\right)\right]. \qquad \ldots(6.2.23)$$

In (6.2.16), putting $x = 0$, replacing y by $\frac{y}{a}$, making $a \to \infty$ and using Kummer's first transformation (5.2.13), we get the following Ramanujan's theorem [88, p.106(5)]

$$ {}_1F_1\left[\begin{matrix} b; \\ \\ e; \end{matrix}\; y\right] {}_1F_1\left[\begin{matrix} b; \\ \\ e; \end{matrix}\; -y\right] = {}_2F_3\left[\begin{matrix} b, e-b; \\ \\ e, \frac{e}{2}, \frac{e+1}{2}; \end{matrix}\; \frac{y^2}{4}\right]. \qquad \ldots(6.2.24)$$

6.3 TRANSFORMATIONS OF $^{(1)}H_3^{(3)}$- FUNCTION :

Consider another particular case of Exton's generalized Horn function $^{(k)}H_3^{(n)}$ (1.12.6) for $k = 1$ and $n = 3$ in the form

$$^{(1)}H_3^{(3)}\left[a, b, c; d; x, y, z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p}\,(b)_n\,(c)_p\, x^m y^n z^p}{(d)_{m+n+p}\; m!\; n!\; p!}$$

$$= \sum_{n,p=0}^{\infty} \frac{(a)_{n+p}(b)_n(c)_p\, y^n\, z^p}{(d)_{n+p}\; n!\; p!}\; {}_2F_1\left[\begin{matrix} \frac{a+n+p}{2}, \frac{a+n+p+1}{2}; \\ \\ d+n+p \quad ; \end{matrix}\; 4x\right]. \qquad \ldots(6.3.1)$$

If we use the result [33, p.65(26); see also 62, p.251(9.6.5)]

$${}_2F_1\left[\begin{matrix} a,\ a+\frac{1}{2}\ ; \\ b\ ; \end{matrix}\ z\right] = 2^{2a}\left[1+(1-z)^{1/2}\right]^{-2a}$$

$$\cdot {}_2F_1\left[\begin{matrix} 2a,\ 2a-b+1\ ; \\ b\ ; \end{matrix}\ \frac{1-(1-z)^{1/2}}{1+(1-z)^{1/2}}\right];\ |\arg(1-z)| < \pi, \quad \ldots(6.3.2)$$

in (6.3.1), we get

$$(1)H_3^{(3)}\left[a, b, c; d; \left(\tfrac{1}{4}-x^2\right), y, z\right] = \left(\frac{2}{1+2x}\right)^a$$

$$\cdot F_D^{(3)}\left[a; a-d+1, b, c; d; \frac{1-2x}{1+2x}, \frac{2y}{1+2x}, \frac{2z}{1+2x}\right], \quad \ldots(6.3.3)$$

where $F_D^{(3)}$ is Lauricella's triple hypergeometric function of second order given by (1.7.3).

Now using a transformation of Carlson [23, p.224(13)]

$$F_D^{(3)}\left[a; b, d, e; c; x, y, z\right]$$

$$= F^{(3)}\left[\begin{matrix} a, b+d+e :: \text{---}; \text{---}; \text{---} : b; d; \text{---}; \\ c :: b+d+e; \text{---}, \text{---} : \text{---}; \text{---}; \text{---}; \end{matrix}\ x-z, y-z, z\right], \quad \ldots(6.3.4)$$

in (6.3.3), we get

$$(1)\ H_3^{(3)}\left[a,\ b,\ c;\ d;\ (\tfrac{1}{4}-x^2),\ y, z\right] = \left(\frac{2}{1+2x}\right)^a .$$

$$F^{(3)}\left[\begin{matrix} a, 1+a+b+c-d :: \text{———} ; \text{—} ; \text{—} : a-d+1;\ b;\ \text{—} ; \\ \\ d \quad :: 1+a+b+c-d ; \text{—} ; \text{—} : \text{———} ; \text{—} ,\ \text{—} ; \end{matrix}\right.$$

$$\left.\frac{1-2(x+z)}{1+2x},\ \frac{2(y-z)}{1+2x},\ \frac{2z}{1+2x}\right]. \qquad \ldots(6.3.5)$$

Replacing $(\frac{1}{4}-x^2)$ by $(\frac{x}{2})-(\frac{x}{2})^2$ and setting $z = o$ or $c = o$, (6.3.3) reduces to

$$H_3\left[a,\ b;\ d;\ (\tfrac{x}{2})-(\tfrac{x}{2})^2,\ y\right] = \left(\frac{2}{2-x}\right)^a$$

$$.\ F_1\left[a;\ a-d+1,\ b;\ d;\ \frac{x}{2-x}, \frac{2y}{2-x}\right], \qquad \ldots(6.3.6)$$

where H_3 is Horn's function given by (1.5.3).

Now replacing a by d-a, x by $\dfrac{x(x-1)}{(1-2x)^2}$ in (6.3.1) and using a result [88, p.70 (Ex.11)]

$${}_2F_1\left[\begin{matrix} a,\ 1-a; \\ \qquad\qquad z \\ c\ ; \end{matrix}\right] = (1-z)^{c-1}\ (1-2z)^{a-c}$$

$$\cdot {}_2F_1\left[\begin{matrix}\frac{c-a}{2}, \frac{c-a+1}{2} ; & \\ & \frac{4z(z-1)}{(1-2z)^2} \\ c ; & \end{matrix}\right] , \qquad \ldots(6.3.7)$$

we get

$$ {}^{(1)}H_3^{(3)}\left[d-a, b, c; d; \frac{x(x-1)}{(1-2x)^2}, y, z\right] = (1-x)^{1-d}(1-2x)^{d-a}$$

$$\cdot F_S\left[a, d-a, d-a; 1-a, b, c; d, d, d; x, \frac{y(1-2x)}{(1-x)}, \frac{z(1-2x)}{(1-x)}\right], \qquad \ldots(6.3.8)$$

where F_S is Lauricella's function in Saran's notation given by the equation (1.7.7).

In (6.3.8), putting $z = 0$ or $c = 0$, replacing d by $a+d$ and y by $\frac{y(1-x)}{(1-2x)}$, we get

$$H_3\left[d, b; a+d; \frac{x(x-1)}{(1-2x)^2}, \frac{y(1-x)}{(1-2x)}\right]$$

$$= (1-x)^{1-a-d}(1-2x)^{d} F_3\left[a, d; 1-a, b; a+d; x, y\right], \qquad \ldots(6.3.9)$$

where F_3 is Appell's function of third kind given by (1.3.3).

The transformation (6.3.9) was earlier obtained by Erdélyi [31, p.382 (8.4)].

In (6.3.1), replacing x by $\dfrac{xt}{(1-x-t)^2}$ and using a result of Manocha (4.2.15), we get

$$F_P^{(4)}\left[\begin{matrix} a,d :: -;\ -;\ -;\ -:\ -;\ b;\ c;\ -; \\ -:: d\ ;\ d\ ;\ -;\ -:\ -;\ -;\ -;\ -; \end{matrix}\ x, y(1-x-t), z(1-x-t), t\right]$$

$$= (1-x-t)^{-a}\ {}^{(1)}H_3^{(3)}\left[a, b, c, d;\ \frac{xt}{(1-x-t)^2}, y, z\right] \quad \ldots(6.3.10)$$

where $F_P^{(4)}$ is Pathan's quadruple hypergeometric function given by the equation (1.13.1).

When $z = 0$ or $c = 0$, (6.3.10) reduces to

$$F^{(3)}\left[\begin{matrix} a,d :: -;\ -\ ;\ -\ :\ -;\ b\ ;\ -\ ; \\ - :: d;\ d\ ;\ -\ :\ -;\ -;\ -\ ; \end{matrix}\ x, y(1-x-t), t\right]$$

$$= (1-x-t)^{-a}\ H_3\left[a, b; d;\ \frac{xt}{(1-x-t)^2}, y\right]. \quad \ldots(6.3.11)$$

Setting $t = 0$ and replacing y by $\dfrac{y}{1-x}$ and z by $\dfrac{z}{1-x}$, (6.3.10) reduces to

$$F_G\left[a, a, a; A, b, c; A, d, d; x, y, z\right]$$

$$= (1-x)^{-a}\ F_1\left[a; b, c; d;\ \frac{y}{1-x}, \frac{z}{1-x}\right], \quad \ldots(6.3.12)$$

where F_G is Lauricella's triple hypergeometric function in Saran's revised notation given by the equation (1.7.6).

Replacing a by 2a+ b+ c-1, d by a+b+c in (6.3.3), we get

$$R\left[2a+b+c-1;\ a,b,c;\ \frac{4x}{1+2x},\ \frac{1+2(x-y)}{1+2x},\ \frac{1+2(x-z)}{1+2x}\right]$$

$$= \left(\frac{2}{1+2x}\right)^{1-2a-b-c} {}^{(1)}H_3^{(3)}\left[2a+b+c-1,b,c;a+b+c;\left(\tfrac{1}{4}-x^2\right),y,z\right], \quad \ldots(6.3.13)$$

where R is Carlson's triple hypergeometric function given by the equation (1.8.1).

It is to be noted that the three standard kinds of elliptic integrals (complete elliptic integral and incomplete elliptic integrals of first and second kinds) are hypergeometric functionsof the type $F_D^{(3)}$. Therefore the function $F_D^{(3)}$ has special importance for applied Mathematics and Mathematical Physics. Although R and $F_D^{(3)}$ are equivalent, R turns out to be more convenient for both theory and applications (see for example, the work of Carlson [18]).

In the last, we add one more transformation connecting Horn's functions H_1 (1.5.2) and G_1 (1.5.1) in the form

$$H_1\left[e-a,\ 1-a,\ a;\ e;\ y,\ x\right] = (1+x)^{a-1} .$$

$$.\ G_1\left[1-a,\ e-a,\ 1-e;\ \frac{-x}{1+x}\ ,\ \frac{-y}{1+x}\right] \quad . \qquad \ldots(6.3.14)$$

Let us put

$$G = G_1\left[1-a,\ e-a,\ 1-e;\ \frac{-x}{1+x}\ ,\ \frac{-y}{1+x}\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{(1-a)_{m+n}\ (e-a)_{n-m}\ (1-e)_{m-n}}{m!\ \ n!} \left(\frac{-x}{1+x}\right)^m \left(\frac{-y}{1+x}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(1-a)_n (e-a)_n}{(e)_n\ \ n!} \left(\frac{y}{1+x}\right)^n {}_2F_1\left[\begin{matrix}1-a+n,\ 1-e-n; & \\ & \frac{x}{1+x} \\ 1-e-n+a\ ; & \end{matrix}\right] . \qquad \ldots(6.3.15)$$

Now using Euler's first linear transformation (4.2.12) in (6.3.15), we get

$$G = (1+x)^{1-a} \sum_{n=0}^{\infty} \frac{(1-a)_n (e-a)_n\ y^n}{(e)_n\ \ n!} \quad {}_2F_1\left[\begin{matrix}1-a+n,\ a; & \\ & -x \\ 1-e-n+a\ ; & \end{matrix}\right]$$

$$= (1+x)^{1-a} \sum_{m,n=0}^{\infty} \frac{(e-a)_{n-m}\ (1-a)_{n+m}\ (a)_m\ y^n\ x^m}{(e)_n\ \ n!\ m!} \ ,$$

which is converted to (6.3.14) with the help of the definition

(1.5.2) of Horn's function H_1.

6.4 REDUCIBILITY OF K_1-FUNCTION :

Expressing K_1-function (1.12.1) in the form

$$K_1\left[a, a, a, a; b, b, b, c; d, e, f, d; x, y, z, t\right]$$

$$= \sum_{n,p=0}^{\infty} \frac{(a)_{n+p}\,(b)_{n+p}\,y^n z^p}{(e)_n\,(f)_p\,n!\,p!}\; F_1\left[a+n+p; b+n+p, c; d; x, t\right]. \qquad \ldots(6.4.1)$$

Putting $t = x$ in (6.4.1) and using a result [7, p.79(1)]

$$F_1\left[a; b, c; d; x, x\right] = {}_2F_1\left[\begin{matrix} a, b+c ; \\ \\ d ; \end{matrix}\; x\right], \qquad \ldots(6.4.2)$$

we get

$$K_1\left[a, a, a, a; b, b, b, c; d, e, f, d; x, y, z, x\right]$$

$$= F^{(3)}\left[\begin{matrix} a, b+c :: - ; b ; - : - ; - ; - ; \\ \\ - :: - ; b+c ; - : d ; e, f ; \end{matrix}\; x, y, z\right]. \qquad \ldots(6.4.3)$$

When $y = 0$, (6.4.3) reduces to a reduction formula for Lauricella's function F_{14} given by (1.7.5)

$$F_p\left[a,\ a,\ a;\ b,\ c,\ b;\ f,\ d,\ d;\ z,\ x,\ x\right]$$

$$= F^{2:0;1}_{0:1;2}\left[\begin{matrix} a, b+c : - ; & b ; \\ \text{———} : d ; f, b+c ; \end{matrix}\ x,\ z\right]. \qquad \ldots(6.4.4)$$

When $f = b$, (6.4.4) reduces to

$$F_p\left[a,\ a,\ a;\ b,\ c,\ b;\ b,\ d,\ d;\ z,\ x,\ x\right]$$

$$= F_4\left[a;\ b+c;\ d,\ b+c;\ x,\ z\right], \qquad \ldots(6.4.5)$$

where F_4 is Appell's function of fourth kind given by (1.3.4).

Setting $z = x$ in (6.4.5) and using a result of Burchnall (4.2.2), we get

$$F_p\left[a,\ a,\ a;\ b,\ c,\ b;\ b,\ d,\ d,\ x,\ x,\ x\right]$$

$$= {}_3F_2\left[\begin{matrix} a,\ \frac{d+b+c}{2},\ \frac{d+b+c-1}{2}; \\ d,\ d+b+c-1 ; \end{matrix}\ 4x\right]. \qquad \ldots(6.4.6)$$

When $z = -x$, $d = b+c$ and using a result of Srivastava [108, p.296(9); see also 16, p.439 (3.7)]

$$F_4\left[a;\ b;\ c,\ c;\ x,\ -x\right] = {}_4F_3\left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}, \frac{b}{2}, \frac{b+1}{2}; \\ c, \frac{c}{2}, \frac{c+1}{2}\ ; \end{matrix}\ -4x^2\right], \qquad \ldots(6.4.7)$$

(6.4.5) reduces to

$$F_P\left[a,\ a,\ a;\ b,\ c,\ b;\ b,\ b+c,\ b+c;\ -x,\ x,\ x\right]$$

$$= {}_2F_1\left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}; \\ b+c\ ; \end{matrix}\ -4x^2\right]. \qquad \ldots(6.4.8)$$

6.5 <u>REDUCIBILITY OF K_2-FUNCTION</u> :

Writing K_2-function (1.12.2) in the form

$$K_2\left[a,\ a,\ a,\ a;\ b,\ b,\ b,\ c;\ d,\ e,\ f,\ g;\ x,\ y,\ z,\ t\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\ (b)_{m+n}\ x^m\ y^n}{(d)_m\ (e)_n\ m!\ n!}\ .$$

$$F_2\left[a+m+n;\ b+m+n,\ c;\ f,\ g;\ z,\ t\right]. \qquad \ldots(6.5.1)$$

Putting $g = c$ in (6.5.1) and using a result [7, p.79(3)]

$$F_2\left[a;b,c;b,e;x,y\right] = (1-x)^{-a}\ {}_2F_1\left[\begin{matrix} a,\ c; \\ e\ ; \end{matrix}\ \frac{y}{1-x}\right], \qquad \ldots(6.5.2)$$

we get

$$K_2\left[a, a, a, a; b, b, b, c; d, e, f, c; x, y, z, t\right]$$

$$= (1-t)^{-a}\, F_C^{(3)}\left[a; b; d, e, f; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}\right]. \qquad \ldots(6.5.3)$$

When x = o or y = o or z = o in (6.5.3), we get a known reduction formula of Srivastava [99, p.71(5.3)] in the form

$$F_E\left[a, a, a; c, b, b; c, d, e; t, x, y\right]$$

$$= (1-t)^{-a}\, F_4\left[a; b; d, e; \frac{x}{1-t}, \frac{y}{1-t}\right]. \qquad \ldots(6.5.4)$$

Similarly K_2-series can be written as in the following form

$$K_2\left[a, a, a, a; b, b, b, c; d, e, f, g; x, y, z, t\right]$$

$$= \sum_{p,q=0}^{\infty} \frac{(a)_{p+q}(b)_p\,(c)_q\, z^p t^q}{(f)_p\,(g)_q\; p!\, q!}\, F_4\left[a+p+q; b+p; d, e; x, y\right]. \qquad \ldots(6.5.5)$$

Putting y = x in (6.5.5) and using a result of Burchnall (4.2.2), we get

$$K_2\left[a,a,a,a;\ b,b,b,\ c;\ d,\ e,\ f,\ g;\ x,\ x,\ z,\ t\right]$$

$$=F^{(3)}\left[\begin{matrix} a :: b;\ -;\ -;\ \frac{d+e}{2},\ \frac{d+e-1}{2};\ -;\ c; \\ \\ - :: -;\ -;\ -;\ d,\ e,\ d+e-1;\ f;\ g; \end{matrix}\ 4x,\ z,\ t\right]. \quad \ldots(6.5.6)$$

6.6 REDUCIBILITY OF K_3- FUNCTION :

Consider K_3-series (1.12.3) in the form

$$K_3\ \left[a,\ a,\ a,\ a,\ b,\ b,\ c,\ c;\ d,\ e,\ e,\ d;\ x,\ y,\ z,\ t\right]$$

$$= \sum_{n,p=0}^{\infty} \frac{(a)_{n+p}\ (b)_n\ (c)_p\ y^n z^p}{(e)_{n+p}\ \ n!\ \ p!}\ F_1\left[a+n+p;b+n,c+p;d;x,t\right]. \quad \ldots(6.6.1)$$

Putting $t = x$, $d = a-b-c+1$ in (6.6.1) and using the Goursat's quadratic transformation [33, p.113(36)]

$$_2F_1\left[\begin{matrix} a,\ b\ ; \\ \\ a-b+1\ ; \end{matrix}\ z\right] = (1+z^{1/2})^{-2a}\ _2F_1\left[\begin{matrix} a,a-b+\frac{1}{2}; \\ \\ 2a-2b+1; \end{matrix}\ \frac{4\ z^{1/2}}{(1+z^{1/2})^2}\right], \quad \ldots(6.6.2)$$

we get a reduction formula of K_3 into Lauricella's function F_B (that is F_G in Saran's revised notation),

$$K_3\left[a,a,a,a;\ b,b,c,c;a-b-c+1,\ e,e,\ a-b-c+1;\ x^2,y,z,x^2\right]$$

$$= (1+x)^{-2a}\, F_G\left[a, a, a; a-b-c+\tfrac{1}{2}, b, c; 2a-2b-2c+1, e, e; \frac{4x}{(1+x)^2}, \frac{y}{(1+x)^2}, \frac{z}{(1+x)^2}\right]. \qquad \ldots(6.6.3)$$

When $c = o$, (6.6.3) reduces to a known transformation of Appell and Kampé de Fériet [7, p.99(13); see also 10, p.766(4) and 4, p.27] in the form

$$F_4\left[a; b; a-b+1, e; x^2, y\right] = (1+x)^{-2a}\,.$$

$$F_2\left[a; a-b+\tfrac{1}{2}, b; 2a-2b+1, e; \frac{4x}{(1+x)^2}, \frac{y}{(1+x)^2}\right]. \qquad \ldots(6.6.4)$$

When $z = o$, (6.6.3) reduces to a reduction formula

$$F_P\left[a, a, a; b, c, b; e, a-b-c+1, a-b-c+1; y, x^2, x^2\right]$$

$$= (1+x)^{-2a}\, F_2\left[a; a-b-c+\tfrac{1}{2}, b; 2a-2b-2c+1, e; \frac{4x}{(1+x)^2}, \frac{y}{(1+x)^2}\right].$$

$$\ldots(6.6.5)$$

6.7 CORRECTION TO A TRANSFORMATION OF EXTON :

In the book of Exton [41, p.127(4.4.7)], a transformation of generalised Horn function ${}^{(k)}H_4^{(n)}$ is given in the form

$$(1+x_1+\ldots+x_k)^a \; {}^{(k)}H_4^{(n)} \left[a, b_{k+1}, \ldots, b_n; c_1, \ldots, c_n; x_1^2, \ldots, x_k^2, x_{k+1}, \ldots, x_n\right] = F_A^{(n-1)} \left[a; c_1-1, \ldots, c_k-1, b_{k+1}, \ldots, b_n; 2c_1-1, \ldots, 2c_n-1, c_{k+1}, \ldots, c_n; \frac{2x_1}{1+x_1+\ldots+x_k}, \ldots, \frac{2x_k}{1+x_1+\ldots+x_k}, \frac{x_{k+1}}{1+x_1+\ldots+x_k}, \ldots, \frac{x_n}{1+x_1+\ldots+x_k}\right], \quad \ldots(6.7.1)$$

provided that, for convergence

$$\left|\frac{2x_1}{1+x_1+\ldots+x_k}\right| + \ldots + \left|\frac{2x_k}{1+x_1+\ldots+x_k}\right| + \left|\frac{x_{k+1}}{1+x_1+\ldots+x_k}\right| + \ldots + \left|\frac{x_n}{1+x_1+\ldots+x_k}\right| < 1.$$

This transformation was an extension of an earlier result of Srivastava and Exton [111].

In fact the above transformation (6.7.1) is incorrect. We shall give the correct form of the above transformation by making use of Laplace integral representation [41, p.104 (3.5.4.5)] for ${}^{(k)}H_4^{(n)}$ in the form

$$I_{10} = \Gamma a \; {}^{(k)}H_4^{(n)}\left[a, b_{k+1}, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n\right]$$

$$= \int_0^\infty e^{-t}\, t^{a-1}\; {}_0F_1\left[\begin{matrix} \text{---}; \\ c_1; \end{matrix}\; x_1 t^2\right] \cdots {}_0F_1\left[\begin{matrix} \text{---}; \\ c_k; \end{matrix}\; x_k\, t^2\right] .$$

$$\cdot {}_1F_1\left[\begin{matrix} b_{k+1}; \\ c_{k+1}; \end{matrix}\; x_{k+1}\, t\right] \cdots {}_1F_1\left[\begin{matrix} b_n; \\ c_n; \end{matrix}\; x_n\, t\right] dt \quad , \ldots(6.7.2)$$

where $\mathrm{Re}(a) > 0$.

In (6.7.2), replacing $x_1, \ldots, x_k$ by $x_1^2, \ldots, x_k^2$, respectively and using the Kummer's second transformation [88, p.126(9)]

$${}_0F_1\left[\begin{matrix} \text{---}; \\ a; \end{matrix}\; y^2\right] = e^{-2y}\; {}_1F_1\left[\begin{matrix} a-\frac{1}{2}; \\ 2a-1; \end{matrix}\; 4y\right] \qquad \text{---} (6.7.3)$$

such that $(2a-1)$ is not an odd integer < 0,

to each ${}_0F_1$'s, we have

$$\Gamma a \; {}^{(k)}H_4^{(n)}\left[a, b_{k+1}, \ldots, b_n; c_1, \ldots, c_n; x_1^2, \ldots, x_k^2, x_{k+1}, \ldots, x_n\right]$$

$$= \int_0^\infty \exp\left\{-(1+2x_1+\ldots+2x_k)t\right\} t^{a-1}\; {}_1F_1\left[\begin{matrix} c_1-\frac{1}{2}; \\ 2c_1-1; \end{matrix}\; 4x_1 t\right] .$$

$$\ldots {}_1F_1\left[\begin{matrix} c_k-\frac{1}{2}; \\ 2c_k-1; \end{matrix}\; 4x_k t\right] \; {}_1F_1\left[\begin{matrix} b_{k+1}; \\ c_{k+1}; \end{matrix}\; x_{k+1} t\right] \cdots {}_1F_1\left[\begin{matrix} b_n; \\ c_n; \end{matrix}\; x_n t\right] dt$$

$$= \frac{\Gamma a}{(1+2x_1+\ldots+2x_k)^a} \; F_A^{(n)}\Big[a; c_1-\tfrac{1}{2},\ldots,c_k-\tfrac{1}{2}, b_{k+1},\ldots,b_n;$$

$$2c_1-1,\ldots,2c_k-1,\; c_{k+1},\ldots,\; c_n;\; \frac{4x_1}{1+2x_1+\ldots+2x_k},\ldots,\; \frac{4x_k}{1+2x_1+\ldots+2x_k},$$

$$\frac{x_{k+1}}{1+2x_1+\ldots+2x_k},\ldots,\; \frac{x_n}{1+2x_1+\ldots+2x_k}\Big]. \qquad \ldots(6.7.4)$$

From (6.7.4), we have the following correct form of Exton's result (6.7.1)

$$(1+2x_1+\ldots+2x_k)^a \; {}^{(k)}H_4^{(n)}\Big[a, b_{k+1},\ldots,b_n; c_1,\ldots,c_n; x_1^2,\ldots,x_k^2,$$

$$x_{k+1},\ldots,x_n \Big] = F_A^{(n)}\Big[a; c_1-\tfrac{1}{2},\ldots,c_k-\tfrac{1}{2}, b_{k+1},\ldots,\; b_n;$$

$$2c_1-1,\ldots,2c_k-1,\; c_{k+1},\ldots,c_n;\; \frac{4x_1}{1+2x_1+\ldots+2x_k},\ldots,\frac{4x_k}{1+2x_1+\ldots+2x_k},$$

$$\frac{x_{k+1}}{1+2x_1+\ldots+2x_k},\ldots,\; \frac{x_n}{1+2x_1+\ldots+2x_k}\Big]. \qquad \ldots(6.7.5)$$

In the last, we add one more transformation for two n-ple series.

Replacing in (6.7.2) $x_{k+1},\dots, x_n, c_{k+1},\dots, c_n$ by $4x_{k+1},\dots, 4x_n, 2b_{k+1},\dots, 2b_n$, respectively and using Kummer's second transformation (6.7.3) to each ${}_1F_1$'s, we have

$$\Gamma a \; {}^{(k)}H_4^{(n)}\left[a, b_{k+1},\dots,b_n; c_1,\dots,c_k, 2b_{k+1},\dots,2b_n; x_1,\dots, x_k, 4x_{k+1},\dots, 4x_n\right]$$

$$= \int_0^\infty e^{-t(1-2x_{k+1}-\dots-2x_n)}\, t^{a-1}\, {}_0F_1\left[\begin{matrix} --- ; \\ c_1 ; \end{matrix}\; x_1 t^2\right] \dots {}_0F_1\left[\begin{matrix} --- ; \\ c_k ; \end{matrix}\; x_k t^2\right] {}_0F_1\left[\begin{matrix} --- ; \\ \frac{1}{2}+b_{k+1} ; \end{matrix}\; x_{k+1}^2 t^2\right] \dots {}_0F_1\left[\begin{matrix} --- ; \\ \frac{1}{2}+b_n ; \end{matrix}\; x_n^2 t^2\right] dt$$

$$= \frac{\Gamma a}{(1-2x_{k+1}-\dots-2x_n)^a}\, F_C^{(n)}\left[\frac{a}{2}, \frac{a+1}{2}; c_1,\dots,c_k, \frac{1}{2}+b_{k+1}, \dots, \frac{1}{2}+b_n; \frac{4x_1}{(1-2x_{k+1}-\dots-2x_n)^2},\dots, \frac{4x_k}{(1-2x_{k+1}-\dots-2x_n)^2},\right.$$

$$\left. \frac{4x_{k+1}^2}{(1-2x_{k+1}-\ldots-2x_n)^2}, \ldots, \frac{4x_n^2}{(1-2x_{k+1}-\ldots-2x_n)^2} \right] . \quad \ldots(6.7.6)$$

From (6.7.6), we have another transformation of ${}^{(k)}H_4^{(n)}$ into Lauricella's function $F_c^{(n)}$ in the form

$$(1-2x_{k+1}-\ldots-2x_n)^a \; {}^{(k)}H_4^{(n)} \Big[a, b_{k+1}, \ldots, b_n; c_1, \ldots, c_k,$$

$$\left. 2b_{k+1}, \ldots, 2b_n; x_1, \ldots, x_k, \; 4x_{k+1}, \ldots, 4x_n \right]$$

$$= F_c^{(n)} \Big[\frac{a}{2}, \frac{a+1}{2}; c_1, \ldots, c_k, \frac{1}{2} + b_{k+1}, \ldots, \frac{1}{2} + b_n;$$

$$\frac{4x_1}{(1-2x_{k+1}-\ldots-2x_n)^2}, \ldots, \frac{4x_k}{(1-2x_{k+1}-\ldots-2x_n)^2}, \frac{4x_{k+1}^2}{(1-2x_{k+1}-\ldots-2x_n)^2}$$

$$\left. \ldots, \frac{4x_n^2}{(1-2x_{k+1}-\ldots-2x_n)^2} \right] . \qquad \ldots(6.7.7)$$

When $k = 1$ and $n = 2$, (6.7.5) reduces to a transformation (6.2.17), which further reduces to a known quadratic transformation [33, p.111(4)] in the form

$${}_2F_1 \left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}; \\ c_1 ; \end{matrix} \; 4x_1^2 \right] = (1+2x_1)^{-a} \; {}_2F_1 \left[\begin{matrix} a, c_1 - \frac{1}{2}; \\ 2c_1 - 1 ; \end{matrix} \; \frac{4x_1}{1+2x_1} \right] .$$

$$\ldots(6.7.8)$$

When k = 1 and n = 2, (6.7.7) reduces to a known transformation (6.2.7) of Erdélyi.

When k = o, (6.7.7) reduces to a known tranformation of Srivastava and Exton [111, p.39(6)]

$$(1-2x_1-\ldots-2x_n)^{a}\ F_A^{(n)}\left[a;b_1,\ldots,b_n;2b_1,\ldots,2b_n;4x_1,\ldots,4x_n\right]$$

$$= F_C^{(n)}\left[\frac{a}{2};\frac{a+1}{2};\frac{1}{2}+b_1,\ldots,\frac{1}{2}+b_n;\frac{4x_1^2}{(1-2x_1-\ldots-2x_n)^2},\ldots,\right.$$

$$\left.\ldots,\frac{4x_n^2}{(1-2x_1-\ldots-2x_n)^2}\right], \qquad \ldots(6.7.9)$$

which holds true when $|x_1| + \ldots + |x_n| < \frac{1}{4}$.

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APPENDIX

ON SOME TRANSFORMATION AND REDUCTION FORMULAE OF GENERALIZED HORN FUNCTION $H_4^{(n)}$—I

BY

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ON SOME TRANSFORMATION AND REDUCTION FORMULAE OF GENERALIZED HORN FUNCTION $H_4^{(P)}$—I

BY

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Abstract. In the present paper, we obtain the correct form of results of Khan and Pathan [9] which appeared in this journal. A number of transformations and reductions of Generalized Horn function of three variables $H_4^{(P)}$, Srivastava's $F^{(3)}$ and their combinations together with their special cases are also discussed.

1. **Introduction.** In a recent paper of Khan and Pathan [9] which appeared in this journal, a triple hypergeometric series $H_4^{(P)}$ [9; p. 85 (1.1)] was considered in the form:

$$H_4^{(P)}[a;\ b,\ c;\ d,\ e,\ f;\ x,\ y,\ z] = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p}\,(b)_n\,(c)_p\, x^m\, y^n\, z^p}{(d)_m\,(e)_n\,(f)_p\, m!\,n!\,p!}, \tag{1.1}$$

where, as usual $(a)_m = \Gamma(a+m)/\Gamma(a)$ and for convergence of multiple power series, we have the following Cartesian equation

$$4r = (s+t-1)^2,\ |x| < r,\ |y| < s,\ \text{and}\ |z| < t$$

where positive quantities r, s, and t are associated radii of convergence.

In fact $H_4^{(P)}$ is a special case of Exton's function ${}^{(k)}H_4^{(n)}$ [8; p. 97 (3.5.2)] when $k = 1$, $n = 3$ and is a generalization of Horn's function H_4[7; p. 225 (16)] and Appell's function of second kind F_2[7: p. 224(7)]. An integral representation for $H_4^{(P)}$ was the main tool used in [9] to obtain transformation and reduction formulae of hypergeometric functions of three variables. A closer examination of reduction formulae of Khan and Pathan [9; p. 90 (4.9, 4.11)] would reveal the fact that these reduction formulae are not correct.

We first obtain in §2, the correct forms of [9; p. 90 (4.9, 4.11)] by using series manipulations and then proceed in §3 to obtain a number of transformations for $H_4^{(P)}$ into a triple series $F^{(3)}$ or their combinations. Some special cases of its reductions to Appell's functions F_2, F_4, Horn's function H_4 and Kampé de Fériet's function are mentioned in §4.

2. **Corrections to reduction formulae of Khan and Pathan.** We have from (1.1)

$$
\begin{aligned}
&H_4^{(P)}[a;\ b,\ c;\ 2d-1,\ 2b,\ 2c;\ -y,\ x,\ -x] \\
(2.1)\qquad &= \sum_{m=0}^{\infty} \frac{(a)_{2m}(-y)^m}{(2d-1)_m m!} \cdot F_2[a+2m;\ b, c;\ 2b, 2c;\ x,\ -x]
\end{aligned}
$$

which on using a result of Bailey [3; p. 239 (4.4)] and interpreting the result with the help of the definition of Kampé de Fériet's double hypergeometric function $F^{p:q;l}_{m:n;s}$ in the notation of Srivastava and Panda [15; p. 423 (26)], gives a correct form of Khan and Pathan [9; p. 90 (4.9)]

$$
\begin{aligned}
&H_4^{(P)}[a;\ b,\ c;\ 2d-1,\ 2b,\ 2c;\ -y, x,\ -x] \\
(2.2)\qquad &= F^{2:0;2}_{0:1;3}\left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2} :\ -;\ \frac{b+c}{2}, \frac{b+c+1}{2}\ ; \\ -:\ 2d-1;\ \frac{2b+1}{2}, \frac{2c+1}{2}, b+c; \end{matrix}\ -4y,\ x^2\right].
\end{aligned}
$$

In (2.2) and in a transformation formula [9; p. 85 (1.2)] of Srivastava's triple hypergeometric series $F^{(3)}$ [13; p. 428; see also 5; p. 40]

$$
\begin{aligned}
&H_4^{(P)}[a;\ c,\ e;\ a-b+1,\ d,\ f;\ -x,\ y,\ z] \\
&= \left(\frac{2}{1+(1+4x)^{1/2}}\right)^a \\
(2.3)\qquad &\cdot F^{(3)}\left[\begin{matrix} a,\ b::\ -;\ -;\ -:\ -\ ;\ c;\ e; \\ -::\ -;\ b;\ -:\ a-b+1;\ d;\ f; \end{matrix}\right. \\
&\left.\frac{1-(1+4x)^{1/2}}{1+(1+4x)^{1/2}}, \frac{2y}{1+(1+4x)^{1/2}}, \frac{2z}{1+(1+4x)^{1/2}}\right],
\end{aligned}
$$

suitably adjusting the variables and parameters and comparing the right hand sides of (2.2) and (2.3), we have another correct form of Khan and Pathan [9; p. 90 (4.11)]

$$F^{(3)}\left[\begin{matrix} a,\ a-2d+2::\text{—}; & \text{——}; & \text{—}: & \text{——}; & b; & c; \\ \text{——}::\text{—}; & a-2d+2; & \text{—}: & 2d-1; & 2b; & 2c; \end{matrix}\ \frac{1-(1+y)^{1/2}}{1+(1+y)^{1/2}}, \frac{2x}{1+(1+y)^{1/2}}, \frac{-2x}{1+(1+y)^{1/2}}\right]$$

$$(2.4)\qquad = \left(\frac{1+(1+y)^{1/2}}{2}\right)^a \cdot F^{2:2;0}_{0:3;1}\left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}: \frac{b+c}{2}, \frac{b+c+1}{2}; & \text{——}; \\ \text{——}: b+\frac{1}{2}, c+\frac{1}{2}, b+c; & 2d-1; \end{matrix}\ x^2, -y\right].$$

3. **More transformation and reduction formulae.** Consider

$$(3.1)\qquad H_4^{(P)}[a;\ b,\ c;\ d,\ e,\ f;\ x,\ y,\ z] = \sum_{n,p=0}^{\infty} \frac{(a)_{n+p}(b)_n(c)_p y^n z^p}{(e)_n(f)_p n!\, p!} \cdot {}_2F_1\left[\begin{matrix} \frac{a+n+p}{2}, \frac{a+n+p}{2}+\frac{1}{2}; \\ d\ ; \end{matrix}\ 4x\right].$$

Now using a result [7; p. 112 (17)] for Gauss's ordinary hypergeometric function ${}_2F_1$[12; p. 45 (1)] in (3.1) and interpreting the result in the form of Srivastava's $F^{(3)}$, we get

$$(3.2)\qquad H_4^{(P)}[a;\ b,\ c;\ d,\ e,\ f;\ x,\ y,\ z] = \left(\frac{1}{1+2(x^{1/2})}\right)^a F^{(3)}\left[\begin{matrix} a::\text{—};\ \text{—};\ \text{—}: & \frac{2d-1}{2}; & b; & c; \\ ::\text{—};\ \text{—};\ \text{—}: & 2d-1; & e; & f; \end{matrix}\ \frac{4(x^{1/2})}{1+2(x^{1/2})}, \frac{y}{1+2(x^{1/2})}, \frac{z}{1+2(x^{1/2})}\right].$$

If we replace x by $(x/2)-(x/2)^2$ in (3.1) and use a result in [7; p. 111 (6)], we get

$$(3.3)\qquad H_4^{(P)}\left[a;\ b,\ c;\ d,\ e,\ f;\ \left(\frac{x}{2}\right)-\left(\frac{x}{2}\right)^2, y, z\right] = \left(\frac{2}{2-x}\right)^a \cdot F^{(3)}\left[\begin{matrix} a,\ a-d+1::\text{—}; & \text{——}; & \text{—}:\text{—}; & b; & c; \\ \text{——}::\text{—}; & a-d+1; & \text{—}: d; & e; & f; \end{matrix}\ \left(\frac{x}{2-x}\right), \left(\frac{2y}{2-x}\right), \left(\frac{2z}{2-x}\right)\right].$$

Next, we consider

$$H_4^{(P)}[a;\ b,\ c;\ d,\ e,\ 2c;\ x,\ y,\ z] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n y^n}{(e)_n n!} H_4[a+n,\ c;\ d,\ 2c;\ x,\ z], \tag{3.4}$$

and use a result [6; p. 382 (7.6), see also 14; p. 42 (22)]. to get

$$\begin{aligned} H_4^{(P)}[a;\ b,\ c;\ d,\ e,\ 2c;\ x,\ y,\ z] &= \left(\frac{2}{2-z}\right)^a \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (2y/(2-z))^n}{(e)_n n!} \\ &\cdot F_4\left[\frac{a+n}{2};\ \frac{a+n+1}{2};\ d,\ \frac{2c+1}{2};\ \frac{16x}{(2-z)^2},\ \left(\frac{z}{2-z}\right)^2\right] \end{aligned} \tag{3.5}$$

where F_4 is Appell's function of fourth kind [7; p. 224 (9)]. Now using an identity

$$\sum_{n=0}^{\infty} A(n) = \sum_{n=0}^{\infty} A(2n) + \sum_{n=0}^{\infty} A(2n+1) \tag{3.6}$$

in (3.5) and interpreting the result in the form of $F^{(3)}$, we get a reduction of $H_4^{(P)}$ into a combination of two $F^{(3)}$'s in the form

$$\begin{aligned} &H_4^{(P)}[a;\ b,\ c;\ d,\ e,\ 2c;\ x,\ y,\ z] = \left(\frac{2}{2-z}\right)^a \\ &\cdot F^{(3)}\left[\begin{matrix} \frac{a}{2},\frac{a+1}{2} :: -;-;-:-;\ \frac{b}{2},\frac{b+1}{2};\ -;\\ - :: -;-;-:\ d;\ \frac{1}{2},\frac{e}{2},\frac{e+1}{2};\ \frac{2c+1}{2}; \end{matrix}\ \frac{16x}{(2-z)^2},\left(\frac{2y}{2-z}\right)^2,\left(\frac{z}{2-z}\right)^2\right] + \frac{aby}{e}\left(\frac{2}{2-z}\right)^{a+1} \\ &\cdot F^{(3)}\left[\begin{matrix} \frac{a+1}{2},\frac{a+2}{2} :: -;-;-:-;\ \frac{b+1}{2},\frac{b+2}{2};\ -;\\ - :: -;-;-:\ d;\ \frac{3}{2},\frac{e+1}{2},\frac{e+2}{2};\ \frac{2c+1}{2}; \end{matrix}\ \frac{16x}{(2-z)^2},\left(\frac{2y}{2-z}\right)^2,\left(\frac{z}{2-z}\right)^2\right]. \end{aligned} \tag{3.7}$$

Now consider the series in the form

$$S = H_4^{(P)}[a;\ b,\ b;\ d,\ 2b,\ 2b;\ x,\ 2y,\ 2z]$$

expressing it in a series of Appell's F_2 and using the results of Bailey [2; p. 11 (3.1), see also 3; p. 239 (4.7)] and Carlson [4; p. 961], we get a transformation of $H_4^{(P)}$ into $F^{(3)}$

$$H_4^{(P)}[a;\ b,\ b;\ d,\ 2b,\ 2b;\ x,\ 2y,\ 2z]$$

$$= \frac{1}{(1-y-z)^a}$$

(3.8)
$$\cdot F^{(3)}\left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2} :: -; & \text{——} & ; -: -;\ b;\ b; \\ \text{——} :: -;\ 2b, & \frac{2b+1}{2} & ; -:\ d;\ -;\ -; \end{matrix}\right.$$

$$\left.\frac{4x}{(1-y-z)^2}, \left(\frac{y+z}{1-y-z}\right)^2, \left(\frac{y-z}{1-y-z}\right)^2\right].$$

Similarly by suitable adjustment of parameters and variables in $H_4^{(P)}$ and using the transformation of Bailey [1; p. 79 (7)], we get

$$H_4^{(P)}[a;\ b,\ e-b;\ d,\ e,\ e;\ x,\ y,\ y]$$

$$= \frac{1}{(1-y)^a}$$

(3.9)
$$\cdot F^{2:0:2}_{0:1:3}\left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2} : -;\ b,\ e-b\ ; \\ \text{——} : d;\ e,\ \frac{e}{2}, \frac{e+1}{2}; \end{matrix} \frac{4x}{(1-y)^2}, \left(\frac{y}{1-y}\right)^2\right].$$

Similarly with the help of transformations [12; p. 66 (2)] [11; p. 688 (line 11)] in (3.1), we have

$$H_4^{(P)}\left[a;\ b,\ c;\ d,\ e,\ f;\ \frac{x}{(1+x)^2},\ y,\ z\right]$$

(3.10)
$$= (1+x)^a \cdot F^{(3)}\left[\begin{matrix} a,\ a-d+1 :: -; & \text{——} & ; -: -;\ b;\ c; \\ \text{——} & :: -;\ a-d+1; & -:\ d;\ e;\ f; \end{matrix}\right.$$

$$\left. x,\ y(1+x),\ z(1+x)\right].$$

$$H_4^{(P)}\left[a;\ b,\ c;\ d,\ e,\ f;\ \frac{xt}{(1-x-t)^2},\ y,\ z\right]$$

(3.11)
$$= (1-x-t)^a K_{10}[a,\ a,\ a,\ a;\ d,\ d,\ b,\ c;$$

$$d,\ d,\ e,\ f;\ x,\ t,\ y(1-x-t),\ z(1-x-t)]$$

where K_{10} is Exton's function [8; p. 78 (3.3.10)].

4. **Some deductions.** When $z=0$ or $c=0$, (3.2) reduces in the form

$$(4.1)\quad \begin{aligned}&(1+2(x)^{1/2})^a H_4[a,\ b;\ d,\ e;\ x,\ y]\\ &= F_2\left[a;\ d-\frac{1}{2},\ b;\ 2d-1, e;\ \frac{4(x)^{1/2}}{1+2(x)^{1/2}},\ \frac{y}{1+2(x)^{1/2}}\right].\end{aligned}$$

When $b=0$ or $y=0$ and $f=c$ in (3.3), we get

$$(4.2)\quad \begin{aligned}&\left(\frac{2-x}{2}\right)^a H_4\left[a,\ c;\ d,\ c;\ \left(\frac{x}{2}\right)-\left(\frac{x}{2}\right)^2, z\right]\\ &= F_4\left[a;\ a-d+1;\ d,\ a-d+1;\ \frac{x}{2-x},\ \frac{2z}{2-x}\right].\end{aligned}$$

When $z=0$, $e=b$, (3.7) reduces to

$$(4.3)\quad \begin{aligned}H_4[a,\ b;\ d,\ b;\ x,\ y] &= F_4\left[\frac{a}{2};\ \frac{a+1}{2};\ d,\ \frac{1}{2};\ 4x,\ y^2\right]\\ &\quad + ay\cdot F_4\left[\frac{a+1}{2};\ \frac{a+2}{2};\ d,\ \frac{3}{2};\ 4x,\ y^2\right].\end{aligned}$$

When $x=0$, $e=b$, in (3.7), we get

$$(4.4)\quad \begin{aligned}&F_2[a;\ b,\ c;\ b,\ 2c;\ y,\ z]\\ &= \left(\frac{2}{2-z}\right)^a\\ &\quad\cdot F_4\left[\frac{a}{2};\ \frac{a+1}{2};\ \frac{1}{2},\frac{2c+1}{2};\ \left(\frac{2y}{2-z}\right)^2,\left(\frac{z}{2-z}\right)^2\right]\\ &\quad + ay\left(\frac{2}{2-z}\right)^{a+1} F_4\left[\frac{a+1}{2};\ \frac{a+2}{2};\ \frac{3}{2},\frac{2c+1}{2};\right.\\ &\quad \left.\left(\frac{2y}{2-z}\right)^2,\left(\frac{z}{2-z}\right)^2\right].\end{aligned}$$

When $z=y$, (3.8) reduces to

$$(4.5)\quad \begin{aligned}&H_4^{(r)}[a;\ b,\ b;\ d,\ 2b,\ 2b;\ x,\ 2y,\ 2y]\\ &= \frac{1}{(1-2y)^a}\\ &\quad\cdot F^{2:0:1}_{0:1:2}\left[\begin{matrix}\frac{a}{2},\frac{a+1}{2}:-; & b & ;\\ \text{---} : d;\ b+\frac{1}{2}, & 2b & ;\end{matrix}\right.\\ &\quad\left.\frac{4x}{(1-2y)^2},\left(\frac{2y}{1-2y}\right)^2\right].\end{aligned}$$

In (3.9), taking $x=0$, y by y/a, $|a|\to\infty$ and using a result [10; p. 48 (§3.5)] and Kummer's first transformation [12; p. 125 (2)], we can get easily Ramanujan's theorem [12; p. 106 (5)].

In a transformation [9; p. 88 (4.2)], using (3.6) we get

$$\begin{aligned}
&F_4\left[a;\ b;\ a-b+1,\ b;\ \frac{1-(1+4x)^{1/2}}{1+(1+4x)^{1/2}},\ \frac{2z}{1+(1+4x)^{1/2}}\right]\\
&=\left(\frac{1+(1+4x)^{1/2}}{2}\right)^a\\
(4.6)\qquad &\cdot F_4\left[\frac{a}{2};\ \frac{a+1}{2};\ \frac{1}{2},\ a-b+1;\ z^2,\ -4x\right]\\
&+az\left(\frac{1+(1+4x)^{1/2}}{2}\right)^a\\
&\cdot F_4\left[\frac{a+1}{2};\ \frac{a+2}{2};\ \frac{3}{2},\ a-b+1;\ z^2,\ -4x\right].
\end{aligned}$$

When $z\to 0$ and replacing y by $y/(1-x-t)$ in (3.11), we get a known reduction formula of Exton [8; p. 116 (4.1.16)]

$$\begin{aligned}
&F_E[a,\ a,\ a,\ b,\ d,\ d;\ e,\ d,\ d;\ y,\ x,\ t]\\
(4.7)\qquad &=(1-x-t)^{-a}H_4\left[a,\ b;\ d,\ e;\ \frac{xt}{(1-x-t)^2},\ \frac{y}{1-x-t}\right]
\end{aligned}$$

where F_E is Saran's function [8; p. 66].

When $z=0$ or $c=0$ and $e=b$ in (3.10), we have

$$\begin{aligned}
(4.8)\qquad &H_4\left[a,\ b;\ d,\ b;\ \frac{x}{(1+x)^2},\ y\right]\\
&=(1+x)^a F_4[a;\ a-d+1;\ d,\ a-d+1;\ x,\ y(1+x)].
\end{aligned}$$

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A NOTE ON THE REDUCIBILITY OF THE TRIPLE HYPERGEOMETRIC FUNCTION F_E

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Introduction

The present note is motivated by Exton's work [7] which contains, among other results, three erroneous reduction formulas [7; pp.66(3.2), (3.5,3.6)] for Lauricella's triple hypergeometric function F_4 of second order (that is, the function F_E [8; p.66] in Saran's notations) defined as

$$F_E[a,a,a,b,c,c;d,e,f;x,y,z]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}\,(b)_m\,(c)_{n+p}\,x^m y^n z^p}{(d)_m\,(e)_n\,(f)_p\,m!\,n!\,p!}, \tag{1.1}$$

where $|x| < r$, $|y| < s$, $|z| < t$, $r + (s^{1/2} + t^{1/2})^2 = 1$ and $(a)_n = \Gamma(a + n)/\Gamma(a)$.

We give corrections to these results of Exton in section 2. Our proofs will not be along the lines followed by Exton, who used the Laplace integral representation [7; p.66(2.2)]. Instead we shall deduce them by series manipulation techniques. Some more linear, quadratic reduction formulae of F_E into Kampé de Fériet's, Horn's and Gauss's functions and a transformation formula of F_E into Srivastava's function, are given in section 3.

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2. Corrections to Exton's reduction formulae

In (1.1), putting $z = y$, expressing F_E in a series form of Appell's double hypergeometric function of fourth kind F_4 [8; p.24 (1.4.4)] and using a result of Burchnall [4; p.101(37)] in conjunction with the definition of Kampé de Fériet's double hypergeometric function in the notation of Srivastava and Panda [11; p.423(26)], we get the following corrected form of the reduction formula of Exton [7; p.66(3.2); see also 8; p.134(4.7.8)]

$$F_E[a,a,a;b,c,c;d,e,f;x,y,y]$$

$$= F^{1\,:\,1;3}_{0\,:\,1;3}\left[\begin{matrix} a : b ; c, & \frac{e+f-1}{2}, & \frac{e+f}{2} ; \\ - : d ; e, & f, & e+f-1 ; \end{matrix}\; x,\ 4y\right]. \tag{2.1}$$

In (2.1), putting $x = -4y$, $d = 2b$, $f = e = c$, using the results of Bailey [3; p.11(3.1)] and Burchnall [4; p.101(37)], we have

$$F_E[a,a,a;b,c,c;2b,c,c;-4y,y,y]$$

$$= {}_4F_3\left[\begin{matrix} \frac{a}{2}, & \frac{a+1}{2}, & \frac{2b+2c-1}{4}, & \frac{2b+2c+1}{4} ; \\ c, & \frac{2b+1}{2}, & \frac{2b+2c-1}{2} & ; \end{matrix}\; 16y^2\right] \tag{2.2}$$

where ${}_4F_3$ is generalized hypergeometric function [8; p.19(1.3.1)]. (2.2) is a corrected form of Exton's result [7; p.67(3.5); see also 8; p.134(4.7.11)].

Again, putting $x = 4y$, $d = 2b$, $f = e = c$ in (2.1) and proceeding on the same lines of (2.2), we get another corrected form of Exton's result [7; p.67(3.6); see also 8; p.134(4.7.12)]

$$F_E[a,a,a;b,c,c;2b,c,c;4y,y,y]$$

$$= (1-4y)^{-a} \cdot {}_4F_3\left[\begin{matrix} \frac{a}{2}, & \frac{a+1}{2}, & \frac{2b+2c-1}{4}, & \frac{2b+2c+1}{4} ; \\ c, & \frac{2b+1}{2}, & \frac{2b+2c-1}{2} & ; \end{matrix}\; \left(\frac{4y}{1-4y}\right)^2\right]. \tag{2.3}$$

3. More formulae

In (1.1) putting $\beta' = [illegible] = \gamma$, replacing y and z by $y(1-z)$ and $z(1-y)$ respectively, expressing $F_{[illegible]}$ in a series form of Appell's F_4 and using a result of Bailey [2; p.42(4.3)], we get a reduction formula of F_E into Kampé de Fériet's function

$$F_E[\alpha, [illegible], \alpha; \beta, [illegible]; x, y(1-z), z(1-y)]$$

$$= [(1-y)(1-z)]^{[illegible]} F^{[illegible]:1;0}_{[illegible]:2;1}\left[\begin{matrix} [illegible], 1+[illegible] : \beta ; - ; \\ - : \beta, 1+\alpha-\gamma ; \gamma ; \end{matrix} \; \frac{x}{(1-y)(1-z)}, \frac{yz}{(1-y)(1-z)}\right]. \tag{3.1}$$

Setting $\beta = [illegible]$ in (3.1) and using a result of Srivastava [10; p.102(1.6)], we get a reduction formula of F_E into Horn's double hypergeometric function of second order H_1 [8; p.5[illegible]]

$$F_E[[illegible]; x, y(1-z), z(1-y)]$$

$$= [(1-y)(1-z)]^{[illegible]} (1-x-y-z)^{[illegible]}$$

$$H_1\left[1+[illegible], [illegible]; \frac{x}{1-x-y-z}, \frac{yz}{1-x-y-z}\right]. \tag{3.2}$$

Again, setting $\beta = [illegible]$ in (3.1), expressing $F_{[illegible]}$ in terms of Gauss's ordinary hypergeometric function ${}_2F_1$ [8; p.15(1.1.2)] and using a result of Bailey [1; p.81(7)] we get an interesting transformation of F_E into Srivastava's triple hypergeometric function of second order $H_{[illegible]}$ [8; p.74(3.1.2)]

$$F_E[[illegible]; x, y(1-z), z(1-y)]$$

$$= (1-z)^{[illegible]} H_{[illegible]}[1+[illegible]; 1+[illegible]; x, y, z]. \tag{3.3}$$

Similarly in a result of Srivastava [9; p.71(5.3)], making suitable adjustment of parameters and variables, using the results of Erdélyi [5; pp.382(7.6), 383(8.5), 381(7.4)], [6; pp.149(21), 150(22)], Bailey [1; p.79(3)] and Srivastava [10; p.102(1.6)], we have the following reduction formulae

$$F_E\left[a,a,a;b,\ \frac{2a+1}{2},\ \frac{2a+1}{2};b,e,f;x,y,\ \frac{x^2}{1-x}\right]$$

$$= (1-x)^a H_4\left[2a,\ \frac{2f-1}{2};\ e,\ 2f-1;\ \frac{y(1-x)}{4},\ 2x\right], \tag{3.4}$$

$$F_E[a,a,a;b,e,e;b,e,1+a-e;x,y(1-x)(1-z),z(1-x)(1-y)]$$

$$= (1-x)^{-a}((1-y)(1-z))^{-e}G_2\left[e,e,1-e,e-a;\ \frac{y}{1-y},\ \frac{z}{1-z}\right] \tag{3.5}$$

$$= (1-x)^{-a}(1-y)^{-e}H_2\left[e-a,e,e,a;e;\ \frac{-y}{1-y},\ -z\right], \tag{3.6}$$

$$F_E[a,a,a;b,e,e;b,2e,1+a-2e;x,2y(1-x)(1-z),z(1-x)(1-2y)]$$

$$= ((1-x)(1-y))^{-a}H_4\left[a,e;\ \frac{2e+1}{2},\ 1+a-2e;\ \left(\frac{y}{2(1-y)}\right)^2,\ \frac{z}{1-y}\right], \tag{3.7}$$

$$F_E[a,a,a;b,e,e;b,e,1+a-e;x,y(1-x)(1-z),z(1-x)(1-y)]$$

$$= ((1-x)(1-y))^{-a}\cdot {}_2F_1\left[\begin{matrix}a,\ e\ ; \\ 1+a-e\ ;\end{matrix}\ \frac{z}{1-y}\right] \tag{3.8}$$

and

$$F_E[e+f-1,e+f-1,e+f-1;b,e,e;b,e,f;x,y,z]$$

$$= (1-x)^{e-e-f+1}(1-x-y-z)^{-e}\cdot G_1\left[e,1-e,1-f;\ \frac{y}{1-x-y-z},\ \frac{z}{1-x-y-z}\right] \tag{3.9}$$

where G_2, H_2 and H_4 are another types of Horn's double hypergeometric functions of second order [8; p.36].

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BULLETIN
INSTITUTE OF MATHEMATICS
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Professor M. A. Pathan
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April 30, 1983

Dear Professor Pathan:

The Editorial Board of the Bulletin of the Institute of Mathematics, Academia Sinica, is happy to accept your paper with the title

"On a Reduction Formula of Kampe de Feriet's Hypergeometric Function of Higher Order"
(joint with F.U. Khan and M.I. Qureshi)

for publication in the Bulletin. The paper is tentatively scheduled for Vol. 12, No. 2 (June, 1984).

Sincerely yours,

Fon-Che Liu
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Aligarh-[illegible]
[illegible]

Dear Dr Pathan

re: "A note on hypergeometric polynomials" by M.I. Qureshi and M. [illegible]

I enclose two referees reports on your paper, one of which suggests some minor modifications which would improve the presentation. Would you please supply a revised version of your paper, incorporating whichever of the amendments that seem to you to be appropriate. Such a manuscript will be accepted without further refereeing.

Yours sincerely

Dr B Davies
Associate Editor

JOURNAL OF THE AUSTRALIAN MATHEMATICAL SOCI[illegible]

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[illegible] M.A. Pathan
Department of Mathematics,
Aligarh Muslim University,
ALIGARH-202 002
India

Dear [illegible]

Re: A note on Hypergeometric polyno[illegible]s

I have received your revised manuscript [illegible] forwarded it to the Editor, whose address ap[illegible] for publication in the normal way.

Yours sinc[illegible]

Brian Dav[illegible]
Associate [illegible]